Revista Eletrônica Paulista de Matemática

ISSN 2316-9664
v. 22, n. 3, dez. 2022
Edição Iniciação Científica

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Electromagnetism, magnetohydrodynamics and the Tokamak mathematics

Eletromagnetismo, magnetohidrodinâmica e a matemática do Tokamak

Resumo

Neste artigo, descreveremos matematicamente algumas propriedades do reator de fusão nuclear denominado Tokamak, a fim de deduzirmos a equação de Grad-Shafranov que modela o equilíbrio do plasma no interior do reator. Para tal, apresentamos aspectos básicos da fusão nuclear, das equações de Maxwell e da magnetohidrodinâmica.

Palavras-chave: Tokamak. Equações de Maxwell. Magnetohidrodinâmica. Equações diferenciais.

Abstract

In this paper, we gave a mathematical description of some properties of the nuclear fusion reactor called Tokamak, in order to derive the Grad-Shafranov equation which models the plasma equilibrium inside the reactor. For that, we presented some basic aspects of nuclear fusion, Maxwell's equations and magnetohydrodynamics.

Keywords: Tokamak. Maxwell's equations. Magnetohydrodynamics. Differential equations.

Artigo recebido em set. 2022 e aceito em dez. 2022

1 Introduction

The Tokamak is a fusion reactor considered one of the most promising clean energy sources, still in the research and experimentation phase. The word *Tokamak* originates from an acronym of the russian expressions ”torodalnaya kamera”and ”magnitnaya katushka”, which translates to ”toroidal chamber and magnetic coils”[1], and denominates an experimental nuclear fusion reactor created in the early 1950s by the soviet physicists Igor Tamm and Andrei Sakharov, from Oleg Lavrentyev’s original idea. The main goal of the Tokamak research and experiments is to build a trustworthy and self sustainable clean energy source, so necessary in these times of energy crisis.

This reactor is basically formed by an immense electromagnet that contains a vacuum chamber in its interior, in which a plasma ring is confined by an intense magnetic field. The electrical current inside the Tokamak is generated by an inductive effect: the plasma is the secondary winding of a transformer. A current pulse is applied in the primary, and the toroidal current is created by electromagnetic induction (Figure 1).

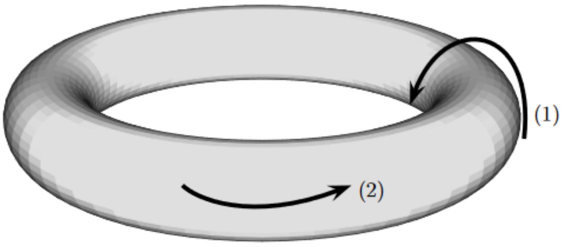

Figura 1: The electrical current flowing in the coils induces a poloidal magnetic field (1) and a toroidal magnetic field (2).

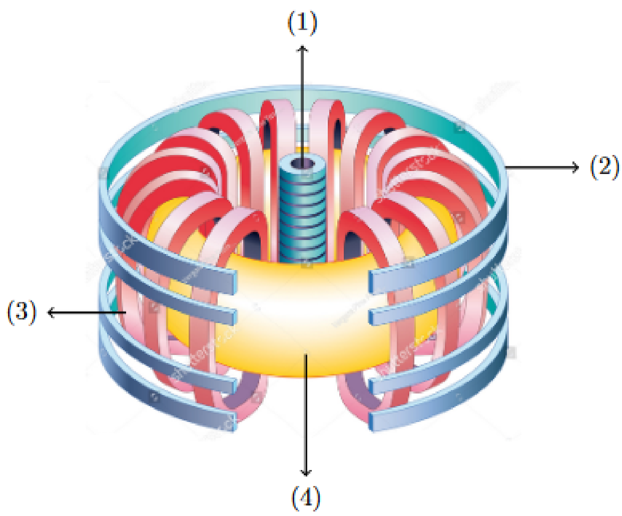

Figura 2: The Tokamak is basically composed of a primary coil (1); position control coils (2); toroidal field coils (3) and the plasma circulating in its interior (4).

The toroidal field coils generate a toroidal magnetic field, and the combination of both of these fields provides the optimal equilibrium of the plasma ring and controls its position inside the vacuum chamber, producing an adequate plasma confinement. The charged plasma particles tend to move along the magnetic field lines and, in the Tokamak, these field lines are helical. Therefore, the particles will remain confined but large scale drifts can arise, for example, in the presence of an electric field perpendicular to the magnetic field. In this case, the particle orbits undergo a drift perpendicular to both fields, which is called $\boldsymbol{E} \times \boldsymbol{B}$ drift (Figure 3).

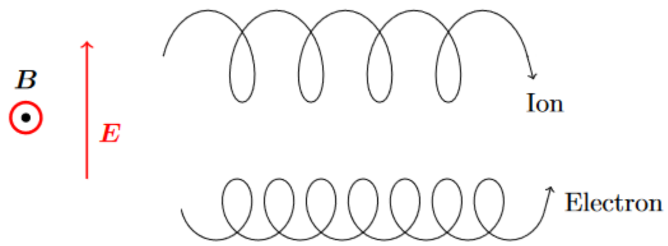

Figura 3: $\boldsymbol{E} \times \boldsymbol{B}$ drift of an ion and an electron.

Another type of drift occurs when a particle is in the presence of a magnetic field with a transverse gradient. The particle orbit has a smaller radius of curvature on the part of its orbit in the stronger magnetic field, and this generates a drift perpendicular to both the magnetic field and its gradient (Figure 4).

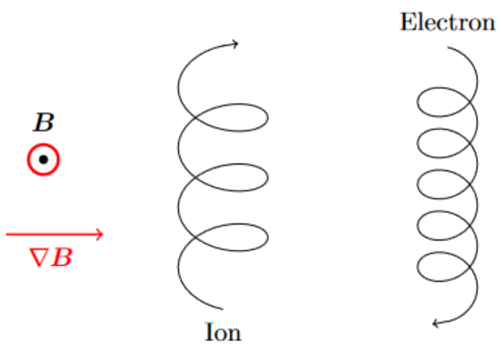

Figura 4: The $\nabla \boldsymbol{B}$ drift: the gradient of $\boldsymbol{B}$ perpendicular to $\boldsymbol{B}$ gives ion and electron drifts in opposite directions.

This system can become incredibly complex of being physically controlled, and the great number of macroscopic and microscopic instabilities causes the mathematical modelling of the plasma equilibrium to be extremely challenging. Since this is a system that involves magnetic fields and electricity, it makes sense to talk about Maxwell's equations. Furthermore, as the plasma behaves like an electrical current conducting fluid in certain situations, it can be described by the magnetohydrodynamics equations. This will lead to the main goal of this paper: to use rigorous mathematical tools to model the ideal equilibrium situation for the plasma inside the Tokamak by using differential equations. As a byproduct, we deduce the Grad-Shafranov equation.

2 The equations of electromagnetism and magnetohydrodynamics

Fundamental in the derivation of the Grad-Shafranov equation, the Maxwell equations are the ones that describe the behaviour and the interaction of the electric and magnetic fields, these being characterized by the system

$$\begin{cases} \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \dfrac{1}{c^2}\dfrac{\partial \boldsymbol{E}}{\partial t}, \\ \nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}, \\ \nabla \cdot \boldsymbol{B} = 0, \\ \nabla \cdot \boldsymbol{E} = \dfrac{\rho_c}{\varepsilon_0}, \end{cases} \tag{1}$$

where $\boldsymbol{B}$ is the magnetic field, $\boldsymbol{E}$ is the electric field, $\boldsymbol{J}$ is the electrical current density, μ_0 is the magnetic permeability constant, ρ_c is the electrical charge density, which can be considered zero due to the quasi-neutrality of the plasma, and c is the speed of light. Also, in this case the displacement current

$$\boldsymbol{J}_d = \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

is negligible in this context. A more detailed approach to the physical aspects inherent to these equations can be found in [2]. In order to find the solutions to Maxwell's equations, notice that from the third equation of (1), we have that

$$\nabla \cdot \boldsymbol{B} = 0 \;\Rightarrow\; \boldsymbol{B} = \nabla \times \boldsymbol{A}, \tag{2}$$

where $\boldsymbol{A}$ is a vector potential. Substituting this expression of $\boldsymbol{B}$ in the second equation of (1) yields

$$\nabla \times \boldsymbol{E} = -\frac{\partial}{\partial t}[\nabla \times \boldsymbol{A}],$$

from which we obtain

$$\nabla \times \left(\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t}\right) = 0.$$

Therefore, the term in parenthesis can be expressed as the gradient of a scalar potential

$$\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t} = -\nabla \phi,$$

from which we get that

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla \phi. \tag{3}$$

Thus, with the view to obtain the expressions of the electric and magnetic fields at (2) and (3), we need to determine the relation between the potentials $\boldsymbol{A}$, ϕ and the fonts ρ_c and $\boldsymbol{J}$. Indeed, considering the fourth equation of (1) and equation (3), we have that

$$\nabla \cdot \left(-\frac{\partial \boldsymbol{A}}{\partial t} - \nabla \phi\right) = \frac{\rho_c}{\varepsilon_0},$$

and so

$$\frac{\partial}{\partial t}[\nabla\cdot\boldsymbol{A}]+\Delta\phi=-\frac{\rho_c}{\varepsilon_0}. \tag{4}$$

Now, we can rewrite the first equation of (1) considering (2) and (3):

$$c^2\big(\nabla\times(\nabla\times\boldsymbol{A})\big)-\frac{\partial}{\partial t}\left[-\nabla\phi-\frac{\partial\boldsymbol{A}}{\partial t}\right]=\frac{\boldsymbol{J}}{\varepsilon_0},$$

or

$$c^2\big(\nabla(\nabla\cdot\boldsymbol{A})-\Delta\boldsymbol{A}\big)+\frac{\partial}{\partial t}[\nabla\phi]+\frac{\partial^2\boldsymbol{A}}{\partial t^2}=\frac{\boldsymbol{J}}{\varepsilon_0}, \tag{5}$$

where $\varepsilon_0=\dfrac{1}{c^2\mu_0}$. As we can always do the gauge transformations

$$\boldsymbol{A}'=\boldsymbol{A}+\nabla\psi\,,\qquad \phi'=\phi-\frac{\partial\psi}{\partial t},$$

for a scalar function ψ without altering the fields $\boldsymbol{E}$ and $\boldsymbol{B}$, then the potentials $\boldsymbol{A}$ and ϕ are not uniquely determined. However, when we take the divergence of $\boldsymbol{A}$ to be the Lorenz gauge

$$\nabla\cdot\boldsymbol{A}=-\frac{1}{c^2}\frac{\partial\phi}{\partial t},$$

we can get a more simplified form of (5):

$$-c^2\Delta\boldsymbol{A}+\frac{\partial^2\boldsymbol{A}}{\partial t^2}=\frac{\boldsymbol{J}}{\varepsilon_0},$$

that is,

$$\Delta\boldsymbol{A}-\frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2}=-\frac{\boldsymbol{J}}{c^2\varepsilon_0}.$$

Moreover, equation (4) becomes

$$\frac{\partial}{\partial t}\left[-\frac{1}{c^2}\frac{\partial\phi}{\partial t}\right]+\Delta\phi=-\frac{\rho_c}{\varepsilon_0},$$

i.e,

$$\Delta\phi-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=-\frac{\rho_c}{\varepsilon_0}.$$

Therefore, if we only know the divergence or the curl of the magnetic or electric field, respectively, it is possible to obtain its expression in terms of a vector potential or a scalar potential. This fact will be applied again later to derive the Grad-Shafranov equation.

Another essential ingredient in obtaining the Grad-Shafranov equation comes from magnetohydrodynamics (MHD), which is a theory used to study the movement of electricity conducting fluids. The MHD equations consists of Maxwell's equations together with the fluid mechanics equations, in which we need to add the effects of the electromagnetic forces. In the present analysis, we will consider the macroscopic behaviour of the plasma, i.e., that it behaves like a single fluid, without making a distinction amongst its particles. In this way, we have the following equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{6}$$

$$\rho \frac{d\boldsymbol{v}}{dt} = \boldsymbol{J} \times \boldsymbol{B} - \nabla p, \tag{7}$$

$$\frac{d}{dt}[p\rho^{-\gamma}] = 0, \tag{8}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{9}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}, \tag{10}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{11}$$

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = 0, \tag{12}$$

where ρ is the mass density, $\boldsymbol{v}$ is the velocity of the fluid and p is the pressure. In addition, $\gamma = 5/3$ is the specific heat ratio and

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla$$

is the material derivative. Equation (6) describes the behaviour of the mass of the system with respect to time and we can see that the total number of particles of the plasma is conserved in the ideal model. The plasma momentum is given by (7), in which we have the inertial force (fictitious), the magnetic force and the pressure gradient interacting in the fluid. In this simplified non-dissipative model, we will assume an adiabatic behaviour (without internal or external heat transfer), which is characterized by equation (8). The equations (9)-(11) are Maxwell's equations.

Finally, we know that the electrical field is equal to zero in a perfectly conducting material, but this rule is not valid when the conductor is moving in a magnetic field. In this case, the *resultant force* in the charges of the material must be zero, because otherwise we would have an infinite flux of free charges. Thus, when we consider the plasma as being perfectly conductive in this ideal MHD model, we conclude that

$$\boldsymbol{F} = q\,(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

and then

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = 0,$$

which is exactly the expression in (12).

3 Nuclear fusion and the plasma movement inside the Tokamak

In the sequel, to be able to model the plasma equilibrium inside the Tokamak we need to understand some of the physical aspects of its functioning. In nuclear fusion reactions, there are two atomic nuclei that combine (or merge) to form a single more massive nucleus. In the Tokamaks, the reaction fuel consists of hydrogen isotopes: deuterium and tritium. When a deuterium nucleus merges with a tritium nucleus, an α-particle is produced and a neutron is released, which results in a reduction of the system's total mass and, consequently, in an energy release by the reaction products according to the equation

$$E = \Delta m c^2,$$

where Δm is the mass variation that occurs during the process. It is possible to have a positive energy variation if the fuel particles react before losing energy. For this to happen, they must retain energy and remain in the reaction region long enough. In temperatures on this scale, the plasma confinement by material walls is impossible, arising the need to use another method. That's where the Tokamak comes in.

Since the deuterium and tritium nuclei inside the Tokamak are positively charged, there is a repulsion between them and therefore the fusion is only achieved if we can manage to overcome this mutual repulsion. For such, these nuclei must have very high speeds and the most promising method to supply this energy is by heating the deuterium-tritium fuel to a sufficiently high temperature: approximately 10 keV $\approx$ 100 million degrees Celsius, at which the fuel becomes completely ionized. This type of fusion if called *thermonuclear fusion* and its result is a gas named *plasma*.

The plasma consists of a gas formed by positive ions and free electrons when completely ionized, and it is also known as the fourth state of matter. However, there are two special properties that set it apart from other gases: the fact that the electric charge density of the ions and electrons in a plasma are almost equal and its intrinsic ability to conduct electrical current. Although the behaviour of the plasma is determined by the motion of the individual particles of the local electromagnetic field, the restrictions that the magnetic field imposes on this movement causes the plasma to have properties of a fluid at certain scales, which we will describe next.

From Lorentz's law and Newton's second law, we have that the equation of motion of a particle of mass m and charge q in a magnetic field is given by

$$m\frac{d\boldsymbol{v}}{dt} = q\,(\boldsymbol{v} \times \boldsymbol{B}), \tag{13}$$

where v is the velocity and $\boldsymbol{B}$ is the magnetic field. For a uniform field $\boldsymbol{B}$ in the z direction, the components of equation (13) are

$$\begin{cases} \dfrac{dv_x}{dt} = \dfrac{qB}{m}v_y = \omega_c v_y, \\ \dfrac{dv_y}{dt} = -\dfrac{qB}{m}v_x = -\omega_c v_x, \\ \dfrac{dv_z}{dt} = 0, \end{cases} \tag{14}$$

where ω_c is a constant that depends on the plasma. From the third equation of (14), we can see that the velocity v_z of the particles along the magnetic field is constant. In addition, we have

$$v_x = -\frac{1}{\omega_c}\frac{dv_y}{dt},$$

from which we obtain

$$v_x = -\frac{1}{\omega_c}\frac{d}{dt}\left[\frac{1}{\omega_c}\frac{dv_x}{dt}\right] = -\frac{1}{\omega_c^2}\frac{d^2 v_x}{dt^2},$$

that is,

$$\frac{d^2 v_x}{dt^2} = -\omega_c^2 v_x. \tag{15}$$

In a similar way, we can obtain that

$$\frac{d^2 v_y}{dt^2} = -\omega_c^2 v_y. \tag{16}$$

Since (15) is a linear second order differential equation, by letting $v_x(t) = e^{\lambda t}$ and using this caracteristic equation, we can find the solution to this equation, namely

$$\begin{aligned} v_x(t) &= c_1 e^{i\omega t} + c_2 e^{-i\omega t} \\ &= c_1\big(\cos(\omega_c t) + i\text{sen}(\omega_c t)\big) + c_2\big(\cos(\omega_c t) - i\text{sen}(\omega_c t)\big). \end{aligned}$$

Thus,

$$v_x(t) = A_1 \cos(\omega_c t) + A_2 \,\text{sen}(\omega_c t),$$

in which $A_1 = c_1 + c_2$ and $A_2 = i(c_1 - c_2)$. By choosing $A_1 = 0$ and $A_2 = v_\perp$ the perpendicular velocity to the (constant) magnetic field, we conclude that

$$v_x = v_\perp \text{sen}(\omega_c t). \tag{17}$$

A similar procedure shows that, by setting $A_1 = v_\perp$ and $A_2 = 0$, the solution of (16) is

$$v_y = v_\perp \cos(\omega_c t). \tag{18}$$

Therefore, since $v_x = dx/dt$ and $v_y = dy/dt$, where $\boldsymbol{x}(t) = \big(x(t), y(t), z(t)\big)$ represents the trajectory of the plasma, we can integrate both sides of equations (17) and (18) with respect to the t variable, and thus we obtain

$$\begin{cases} x = -\dfrac{v_\perp}{\omega_c}\cos(\omega_c t) = -R_L \cos(\omega_c t), \\ y = \dfrac{v_\perp}{\omega_c}\text{sen}(\omega_c t) = R_L \,\text{sen}(\omega_c t). \end{cases} \tag{19}$$

In the equations above, we have that $R_L = \dfrac{v_\perp}{\omega_c} = \dfrac{v_\perp m}{qB}$ is the *Larmor radius*, which is the transverse radius of the helical orbit of the circular motion of these charged particles in a magnetic field (Figure 5). So, for scales larger than the Larmor radius, the plasma has properties of a fluid. This circular motion is described by the equations in (19) and the particles have constant velocity in the z direction of the magnetic field.

4 The Grad-Shafranov equation

Bearing in mind the previous explanations, we first observe that, since the Tokamak is a compact toroidal surface, in order to obtain the Grad-Shafranov equation we will consider r, θ, z the usual cylindrical coordinates and $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_z$ the corresponding unit vectors of the orthonormal system. Also, as the plasma travels in a helical orbit, we shall assume an axial symmetry in the toroidal direction, that is,

$$\frac{\partial F}{\partial \theta} = 0,$$

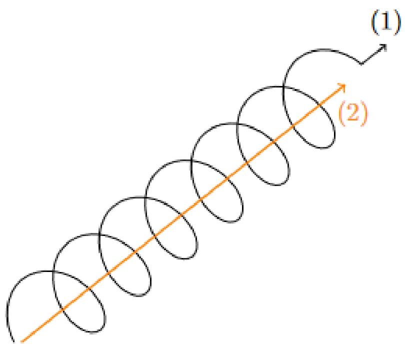

Figura 5: The Larmor radius is the transverse radius of the helical orbit (1) of the particle motion in a magnetic field (2).

for any function F. Here, we will use the equations

$$\begin{cases} \nabla \cdot \boldsymbol{B} = 0, \\ \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}, \\ \boldsymbol{J} \times \boldsymbol{B} = \nabla p \end{cases} \tag{20}$$

of the ideal MHD model in equilibrium ($d\boldsymbol{v}/dt = \boldsymbol{0}$) to obtain the Grad-Shafranov equation. Indeed, since $\nabla \cdot \boldsymbol{B} = 0$, considering a vector potential $\boldsymbol{A} = (A_r, A_\theta, A_z)$, we saw in (2) that

$$\begin{aligned} \boldsymbol{B} = \nabla \times \boldsymbol{A} &= \left(\frac{\partial}{\partial r}\, \boldsymbol{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\, \boldsymbol{e}_\theta + \frac{\partial}{\partial z}\, \boldsymbol{e}_z \right) \times (A_r\, \boldsymbol{e}_r + A_\theta\, \boldsymbol{e}_\theta + A_z\, \boldsymbol{e}_z) \\ &= \boldsymbol{e}_r \times \frac{\partial}{\partial r}(A_r\, \boldsymbol{e}_r + A_\theta\, \boldsymbol{e}_\theta + A_z\, \boldsymbol{e}_z) + \frac{1}{r}\, \boldsymbol{e}_\theta \times \frac{\partial}{\partial \theta}(A_r\, \boldsymbol{e}_r + A_\theta\, \boldsymbol{e}_\theta + A_z\, \boldsymbol{e}_z) \\ &+ \boldsymbol{e}_z \times \frac{\partial}{\partial z}(A_r\, \boldsymbol{e}_r + A_\theta\, \boldsymbol{e}_\theta + A_z\, \boldsymbol{e}_z) \\ &= -\frac{\partial A_\theta}{\partial z}\, \boldsymbol{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \boldsymbol{e}_\theta + \frac{1}{r}\left(A_\theta + r\frac{\partial A_\theta}{\partial r} \right) \boldsymbol{e}_z, \end{aligned}$$

from where we get

$$\boldsymbol{B} = \nabla \times (A_\theta \boldsymbol{e}_\theta) + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \boldsymbol{e}_\theta. \tag{21}$$

Consider the following quantities which characterizes magnetic surfaces:

$$\begin{cases} \nabla \Theta = \dfrac{1}{r}\, \boldsymbol{e}_\theta, \\ \psi = -r A_\theta, \\ f = r\left(\dfrac{\partial A_r}{\partial z} - \dfrac{\partial A_z}{\partial r} \right). \end{cases}$$

With the expressions above, we can write (21) as

$$\boldsymbol{B} = (\nabla\Theta \times \nabla\psi) + f\nabla\Theta. \tag{22}$$

So, the current density is expressed as

$$\begin{aligned}
\mu_0 \boldsymbol{J} = \nabla \times \boldsymbol{B} = \nabla \times \left[(\nabla\Theta \times \nabla\psi) + f\nabla\Theta\right] \\
&= \nabla \times (\nabla\Theta \times \nabla\psi) + \nabla \times (f\nabla\Theta) \\
&= \nabla \times \left[\frac{1}{r}\boldsymbol{e}_\theta \times \left(\frac{\partial\psi}{\partial r}\boldsymbol{e}_r + \frac{\partial\psi}{\partial z}\boldsymbol{e}_z\right)\right] + f\,(\nabla \times \nabla\Theta) + (\nabla f \times \nabla\Theta) \\
&= \nabla \times \left(\frac{1}{r}\frac{\partial\psi}{\partial z}\boldsymbol{e}_r - \frac{1}{r}\frac{\partial\psi}{\partial r}\boldsymbol{e}_z\right) + (\nabla f \times \nabla\Theta) \\
&= \left(\frac{1}{r}\frac{\partial^2\psi}{\partial z^2} - \frac{1}{r^2}\frac{\partial\psi}{\partial r} + \frac{1}{r}\frac{\partial^2\psi}{\partial r^2}\right)\boldsymbol{e}_\theta + (\nabla f \times \nabla\Theta),
\end{aligned}$$

i.e,

$$\mu_0 \boldsymbol{J} = \left(\frac{\partial^2\psi}{\partial z^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial r^2}\right)\nabla\Theta + (\nabla f \times \nabla\Theta), \tag{23}$$

Defining the elliptic differential operator Δ^* as

$$\Delta^* := \frac{\partial^2}{\partial z^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2\psi}{\partial r^2},$$

we can write (23) in the form

$$\mu_0 \boldsymbol{J} = \Delta^*\psi\nabla\Theta + (\nabla f \times \nabla\Theta). \tag{24}$$

On the other hand, from the third equation of (20) and the fact that $d\boldsymbol{v}/dt = 0$, it follows from equation (7) that

$$\nabla p = \boldsymbol{J} \times \boldsymbol{B},$$

from which, assuming $\mu_0 = 1$, we obtain that

$$\begin{cases} \boldsymbol{B} \cdot \nabla p = \boldsymbol{B} \cdot (\boldsymbol{J} \times \boldsymbol{B}) = 0, \\ \boldsymbol{J} \cdot \nabla p = \boldsymbol{J} \cdot (\boldsymbol{J} \times \boldsymbol{B}) = 0. \end{cases} \tag{25}$$

The equations above show that the field lines of the magnetic induction and of the current density lie on isobaric surfaces, that is, surfaces where the pressure is constant. As a result of this fact, these surfaces are also called magnetic surfaces. The plasma boundary is then defined as the outermost closed magnetic surface entirely contained in the vacuum vessel of the Tokamak and, for the optimal plasma confinement, these surfaces must have a toroidal shape. Thus, by using (22) and (25), we get the following relations:

$$\begin{aligned}\boldsymbol{B}\cdot\nabla p &= \left[(\nabla\Theta\times\nabla\psi)+f\nabla\Theta\right]\cdot\nabla p\\ &=\left(\frac{1}{r}\boldsymbol{e}_\theta\times\nabla\psi\right)\cdot\nabla p+(B_\theta\boldsymbol{e}_\theta\cdot\nabla p)\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{J}\cdot\nabla p &= \left[\Delta^*\psi\nabla\Theta+(\nabla f\times\nabla\Theta)\right]\cdot\nabla p\\ &=(\Delta^*\psi\nabla\Theta\cdot\nabla p)+\left(\nabla f\times\frac{1}{r}\boldsymbol{e}_\theta\right)\cdot\nabla p,\end{aligned}$$

from which we conclude that

$$\boldsymbol{B}\cdot\nabla p=\frac{1}{r}\boldsymbol{e}_\theta\cdot(\nabla\psi\times\nabla p)=0 \tag{26}$$

and

$$\boldsymbol{J}\cdot\nabla p=\frac{1}{r}\boldsymbol{e}_\theta\cdot(\nabla f\times\nabla p)=0. \tag{27}$$

Since we assumed an axial symmetry, we have that $\nabla\psi$ and ∇p are perpendicular to $\boldsymbol{e}_\theta$. Then, from equation (26) we get that

$$\nabla\psi\times\nabla p=\mathbf{0},$$

i.e, the vectors $\nabla\psi$ and ∇p are parallel. Similarly for equation (27), we have that ∇f and ∇p are perpendicular to $\boldsymbol{e}_\theta$ and the identity

$$\nabla f\times\nabla p=\mathbf{0}$$

implies that the vectors ∇f and ∇p are parallel, from where we obtain that ∇f and $\nabla\psi$ are also parallel. Therefore, the pressure p and the function f depend only on the poloidal flux, that is, $p=p(\psi)$ and $f=f(\psi)$. Thus, applying the chain rule we can write

$$\nabla p=\nabla\psi\,\frac{dp}{d\psi}\quad \text{e}\quad \nabla f=\nabla\psi\,\frac{df}{d\psi}. \tag{28}$$

Substituting the expressions of $\boldsymbol{B}$ and $\boldsymbol{J}$ obtained in (21) and (23) in the third equation of (20) and considering (28), we obtain

$$\begin{aligned}\nabla\psi\,\frac{dp}{d\psi} &= \left[\Delta^*\psi\nabla\Theta+\left(\nabla\psi\,\frac{df}{d\psi}\times\nabla\Theta\right)\right]\times\left[(\nabla\Theta\times\nabla\psi)+f\nabla\Theta\right]\\ &=(\Delta^*\psi\nabla\Theta)\times(\nabla\Theta\times\nabla\psi)+\left(\nabla\psi\,\frac{df}{d\psi}\times\nabla\Theta\right)\times(f\nabla\Theta)\\ &=\frac{1}{r^2}\,(\Delta^*\psi\,\boldsymbol{e}_\theta)\times(\boldsymbol{e}_\theta\times\nabla\psi)+\frac{f}{r^2}\frac{df}{d\psi}\left[(\nabla\psi\times\boldsymbol{e}_\theta)\times\boldsymbol{e}_\theta\right],\end{aligned}$$

that is,

$$\nabla\psi\,\frac{dp}{d\psi} = -\frac{1}{r^2}\Delta^*\psi\nabla\psi - \frac{f}{r^2}\frac{df}{d\psi}\nabla\psi.$$

Hence, ψ is a solution of the differential equation

$$\Delta^*\psi = -r^2\,\frac{dp}{d\psi} - f\,\frac{df}{d\psi}. \tag{29}$$

The equation (29) is a nonlinear second order partial differential equation called *Grad-Shafranov equation*, which describes the toroidal equilibrium of the plasma inside the Tokamak. This equation can be solved when the functions $p(\psi)$, $f(\psi)$ and the boundary conditions have been assigned. It is typical to search for numerical solutions of equation (29), but there exist some known analytic solutions of the Grad-Shafranov equation [3], [4], [5]. In [5], for example, analytical solutions of the homogeneous Grad-Shafranov equation were obtained by using Green's functions.

5 Concluding remarks

In this paper, we showed how Maxwell's equations and magnetohydrodynamics relate to describe the motion of a plasma inside the Tokamak, which behaves like a fluid in certain scales. From this study, we also presented an alternative way to derive the Grad-Shafranov equation using the potentials associated with the electric and magnetic fields, instead of searching for the explicit expressions of $\boldsymbol{E}$ and $\boldsymbol{B}$.

Currently, there are over 20 active Tokamaks in several countries, the largest of them being the Joint European Torus (JET), located in Culham, England. This Tokamak was the first to successfully produce plasma in 1983, reaching a temperature higher than anywhere in our solar system. There are also a few Tokamak experiments in Brazil: the middle size Tokamak à Chauffage Alfvén Brésilien (TCABR) at the Physics Institute of University of São Paulo and the Experimento Tokamak Esférico (ETE) at the National Institute for Space Research.

One of the most important experimental programs nowadays is the International Thermonuclear Experimental Reactor: the ITER Tokamak. This 10 billion euro project that has the support of countries like China, Japan, England and the United States, is under construction in France and it is expected to be able to produce its first plasma in 2025. The expectation is that this device will be capable of exploring the operation modes of an advanced Tokamak, characterized by high pressure, long plasma confinement periods and sufficient conditions to maintain a self sustainable fusion reaction.

6 Referências

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