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Fractional derivatives as weighted average of historical values: an application to COVID-19 in Brazil

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Abstract

The memory effect is an interesting tool that can be seen in fractional differential equations. To show this clearly, in this paper we prove that the Caputo derivative of a function f, as well as the Riemann-Liouville integral and derivative, are proportional to a weighted average of the historical values of f or f'. For this, we use the statistical expectation of functions, whose random variable follows a beta distribution. Moreover, through the respective probability density functions, for each operator we specify the weight of the historical values of the function to determine its current value, according to the values of the fractional order of the derivative. Furthermore, to prove the effectiveness of the memory effect to describe real phenomena, we compared a classic model with its fractional version to model COVID-19 in Brazil.

Keywords: Memory effect. Caputo derivative. Riemann-Liouville derivative. Statistical expectation. COVID-19.

1 Introduction

Fractional calculus is an area that has attracted more and more researchers, as it generalizes the classical calculus, of integer order. Fractional differential equations have been shown to be very efficient in the mathematical modeling of real phenomena, presenting an interesting feature: the memory effect (BARROS et al., 2021; DIETHELM, 2010; LOPES; SANTOS, 2019; SAEEDIAN et al., 2017).

Among several definitions for fractional operators, the Riemann-Liouville integral and derivative and the Caputo derivative stand out (CAMARGO; OLIVEIRA, 2015). There are a few approaches to showing the memory effect coming from these operators. For example, let f be a real function defined on $[0,t]$ and $t_1, t_2 \in [0,t]$ such that $t_1 < t_2$. Thus, if $H = (J^\alpha f)(t_2) - (J^\alpha f)(t_1)$, where J represents the fractional integral operator, in (BARROS et al., 2021) it is shown that

$$H = \frac{1}{\Gamma(\alpha)}\left[\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f(s)ds + \int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]f(s)ds\right], \text{ for } \alpha \in (0,1),$$

and

$$H = \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f(s)ds = \int_{t_1}^{t_2} f(s)ds, \text{ for } \alpha = 1.$$

That is, when $\alpha = 1$ (classic case), H depends only on what happens in the range $[t_1, t_2]$. By the other side, when $\alpha \in (0,1)$, H depends on what happens in the entire interval $[0, t_2]$, that is, it depends on all historical values, characterizing the effect of memory.

On the other hand, in this paper a statistical approach is used to detect the memory effect on fractional operators, where mathematical expectation is used. More specifically, a weighted average, to explicitly show the memory effect. Moreover, as we indicate in (BARROS et al., 2021), such an effect is characteristic of what is called the phenomenon of hysteresis. This paper was first published in (BARROS et al., 2021) by Springer Nature, but with application made to other countries. Here, we use the SIR epidemiological model to study the spread of COVID-19 in Brazil and use a different approach to determine the recovery rate.

This paper is organized as follows. In Section 2 we present some definitions for a better understanding of the content of this article. The main results are in Section 3 and the application to COVID-19 in Section 4.

2 Preliminary

In this section we present the main concepts of fractional calculus and statistics, for a better understanding of this article.

Definition 1 (Fractional Integral of Riemann-Liouville) (TEODORO; OLIVEIRA; OLIVEIRA, 2018) Let $\alpha \in \mathbb{R}^+, b > 0$ and $f \in L^p([0,b] : \mathbb{R}^m)$, with $1 \le p \le \infty$. The fractional integral of Riemann-Liouville, for $t \in [0,b]$, of order α, is given by

$$J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds. \tag{1}$$

When $\alpha \in \mathbb{N}$, we have $\Gamma(\alpha) = \alpha!$ In this case, Equation (1) becomes the Cauchy formula for iterated integrals, which is the motivation for this definition. Furthermore, for $\alpha > 0$ the integral exists for almost every t in $[0,b]$.

We denote by $AC^n[0,b]$ the set of functions in which the order derivative $n-1$ is absolutely continuous in $[0,b]$ (DIETHELM, 2010).

Definition 2 (Fractional Derivative of Riemann-Liouville) (TEODORO; OLIVEIRA; OLIVEIRA, 2018) Let $\alpha \in \mathbb{R}^+, b > 0, f \in AC^n[0,b]$ and $n = \lfloor \alpha \rfloor$. The fractional derivative of Riemann-Liouville of order α is given by

$$D_t^\alpha f(t) = D_t^n J_t^{n-\alpha} f(t) = \frac{d^n}{dt^n}\left(\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1} f(s)ds\right). \tag{2}$$

Definition 3 (Fractional Derivative of Caputo) (LOPES; SANTOS, 2019; TEODORO; OLIVEIRA; OLIVEIRA, 2018) Let $\alpha \in \mathbb{R}^+, b > 0$ and $f \in AC^n[0,b]$. For $t \in [0,b]$, the fractional derivative of Caputo of order α is given by

$$cD_t^\alpha f(t) \quad = \quad D_t^\alpha (f(t) - f(0)). \tag{3}$$

For $\alpha \in (0,1)$, one can show that

$$cD_t^\alpha f(t) = J_t^{1-\alpha} f'(t). \tag{4}$$

Definition 4 (Beta function) (LOPES; SANTOS, 2019) The beta function is defined by the integral

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx, \text{ where } p,q > 0. \tag{5}$$

Next, we present some important statistics concepts for the development of the work.

Definition 5 (Beta Distribution) (MOOD; GRAYBILL; BOES, 1950) A random variable X follows the beta distribution if its probability density function is given by:

$$f_X(x) = f_X(x;p,q) = \frac{1}{B(p,q)} x^{p-1}(1-x)^{q-1} I_{(0,1)}(x), \tag{6}$$

where $p,q > 0$ and $I_{(0,1)}$ is the indicator function of interval $(0,1)$, that is,

$$I_{(0,1)}(x) = \begin{cases} 1, & \text{if } x \in (0,1) \\ 0, & \text{if } x \notin (0,1) \end{cases}.$$

Definition 6 (Uniform Distribution) (MOOD; GRAYBILL; BOES, 1950) A random variable X follows the uniform distribution over the interval $[a,b]$ if its probability density function is given by:

$$f_X(x) = f_X(x;a,b) = \frac{1}{b-a} I_{[a,b]}(x), \tag{7}$$

for $a,b \in \mathbb{R}$ and $I_{(0,1)}$ is the indicator function of interval $[a,b]$.

Remark 1 *When $p = q = 1$ in* (6)*, the beta distribution coincides with the uniform distribution in the interval* $(0,1)$*, that is, when $a = 0$ and $b = 1$ in* (7).

The expectation or expected value represents the mean of random variable X. Below, we present the definition of this concept for continuous random variable.

Definition 7 (MOOD; GRAYBILL; BOES, 1950) Let X be a continuous random variable with the probability density function $f_X(\cdot)$. The expectation or expected value of X is given by

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)dx. \tag{8}$$

Proposition 1 (MOOD; GRAYBILL; BOES, 1950) Let $g : \mathbb{R} \to \mathbb{R}$ and X be a continuous random variable with the probability density function $f_X(\cdot)$. The expectation or expected value of $g(X)$ is given by

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)dx. \tag{9}$$

In the following section, we propose a proposition with the objective of showing, with a statistical approach, how the memory effect is present in the operators: integral and derivative of Riemann-Liouville and derivative of Caputo.

3 Results on fractional operators

In this section we show that fractional operators applied to a function f can be interpreted as a mathematical expectation of f and/or f', weighted by beta distributions to parameters depending on the order of the derivative. Objectively, this result is illustrated in the following proposition, whose proof can be found in (BARROS et al., 2021).

Proposition 2 *Let $\alpha \in \mathbb{R}^+$ and $f \in AC[0, b]$. Under these conditions, we have*

$$J_t^\alpha f(t) = \frac{t^\alpha}{\Gamma(\alpha+1)} E[f(tU)]; \tag{10}$$

$$D_t^\alpha f(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} E[f(tW)] + \frac{t^{1-\alpha}}{\Gamma(3-\alpha)} E\left[f'(tV)\right], \ \text{if } 0 < \alpha < 1; \tag{11}$$

$$cD_t^\alpha f(t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} E\left[f'(tW)\right], \ \text{if } 0 < \alpha < 1, \tag{12}$$

where U, V and W are random variables with the distributions $U \sim B(1, \alpha), V \sim B(2, 1-\alpha)$ and $W \sim B(1, 1-\alpha)$.

Remark 2 *When $\alpha = 1$ we have $U \sim B(1, 1)$, that is, in the classic case U has a uniform distribution.*

Remark 3 *By applying Definition 3 and* (4) *to equation* (11)*, we get*

$$cD_t^\alpha f(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} E[f(tW) - f(0)] + \frac{t^{1-\alpha}}{\Gamma(3-\alpha)} E\left[f'(tV)\right], \tag{13}$$

which coincides with equation (12)*.*

Remark 4 (BARROS et al., 2021) Note that, for $0 < \alpha < 1, cD^\alpha f(t_\alpha) = 0$ does not imply t_α is a maximum (or minimum) point of f. In fact, suppose that f has just only local maximum in t^*. In this case, we have $f'(s) > 0$ for all $s < t^*$. This implies that $E[f'(tW)] > 0$ for all $t \leq t^*$. Thus, by (12), if $cD^\alpha f(t_\alpha) = 0$ then $t_\alpha > t^*$. That is, $cD^\alpha f(t_\alpha) = 0$ but t_α is not the maximum point of f.

Next, we provide some examples of Riemann-Liouville and Caputo fractional derivatives using the formulas (11) and (12), for the case $\alpha = \frac{1}{2}$.

Example 1 *If* $f(t) = 1$, *we have* $E[f(tW)] = 1$ *and* $E[f'(tV)] = E[f'(tW)] = E[0] = 0$. *Thus, from* (11) *we have* $D_t^{\frac{1}{2}} f(t) = \dfrac{t^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} = \dfrac{t^{-\frac{1}{2}}}{\sqrt{\pi}}$, *and from* (12) *we have* $cD_t^{\frac{1}{2}} f(t) = 0$.

For the following example it is worth noting that $\Gamma(0,5) = \sqrt{\pi}$, $\Gamma(1,5) = \frac{\sqrt{\pi}}{2}$, $\Gamma(2,5) = \frac{3\sqrt{\pi}}{4}$ and $B\left(2, \frac{1}{2}\right) = \frac{4}{3}$.

Example 2 *If* $f(t) = t$, *we have*

$$\begin{aligned} E[f(tW)] &= \int_0^1 \frac{(1-w)^{-\alpha}}{B(1,1-\alpha)} t w dw \\ &= ta(1-\alpha) \int_0^1 (1-w)^{-\alpha} w dw \\ &= t(1-\alpha) B(2, 1-\alpha). \end{aligned}$$

For $\alpha = \frac{1}{2}$,

$$E[f(tW)] = \frac{2t}{3}. \tag{14}$$

Also,

$$\begin{aligned} E[f'(tV)] &= \int_0^1 \frac{(1-v)^{-\alpha} v}{B(2,1-\alpha)} dv \\ &= \frac{1}{B(2,1-\alpha)} \int_0^1 (1-v)^{-\alpha} v dv \\ &= \frac{1}{B(2,1-\alpha)} B(2,1-\alpha) \\ &= 1. \end{aligned}$$

Finally,

$$\begin{aligned} E[f'(tW)] &= \int_0^1 \frac{(1-w)^{-\alpha}}{B(1,1-\alpha)} dw \\ &= \frac{1}{B(1,1-\alpha)} \int_0^1 (1-w)^{-\alpha} dw \\ &= \frac{1}{B(1,1-\alpha)} B(1,1-\alpha) \\ &= 1 \end{aligned}$$

From (11), *we have*

$$\begin{aligned} D_t^{\frac{1}{2}} f(t) &= \frac{2t^{1-\frac{1}{2}}}{3\Gamma\left(\frac{1}{2}\right)} + \frac{t^{\frac{1}{2}}}{\Gamma(3-\frac{1}{2})} \\ &= \frac{2t^{\frac{1}{2}}}{3\sqrt{\pi}} + \frac{4t^{\frac{1}{2}}}{3\sqrt{\pi}} \\ &= \frac{2t^{\frac{1}{2}}}{\sqrt{\pi}}. \end{aligned} \tag{15}$$

From (12)*, we obtain*

$$\begin{aligned} cD_t^{\frac{1}{2}} f(t) &= \frac{t^{\frac{1}{2}}}{\Gamma(2-\frac{1}{2})} \\ &= \frac{2t^{\frac{1}{2}}}{\sqrt{\pi}}. \end{aligned} \tag{16}$$

The results of Examples 1 and 2 coincide with those presented in the literature, showing that the equations (11) and (12) can be used instead of the traditional approach to fractional calculus. Moreover, in Example 2, the equations (15) and (16) are the same, which is correct since $D_t^\alpha f(t) = cD_t^\alpha f(t)$ for $f(0) = 0$ and for all $\alpha \in (0, 1)$ (see Equation (13)).

We saw in (10)-(12) that such operators are proportional to the weighted average of f and/or f', through statistical expectation. The random variables involved follow the beta distribution for different parameters, showing that the values of each one have different weights in the weighted average in question. In the next section we see more details about these weights.

3.1 The weight in each weighted average

In this section we analyze the behavior of the probability density functions of the random variables involved in the equations (10)-(12), for some values of α. As $U \sim B(1, \alpha), V \sim B(2, 1-\alpha)$ and $W \sim B(1, 1-\alpha)$, then the respective probability density functions are given by

$$f_U(u) = \frac{(1-u)^{\alpha-1}}{B(1,\alpha)}, \quad f_V(v) = \frac{v\,(1-v)^{-\alpha}}{B(2,1-\alpha)} \quad \text{and} \quad f_W(w) = \frac{(1-w)^{-\alpha}}{B(1,1-\alpha)}. \tag{17}$$

When $\alpha > 1$, the function f_U is decreasing. For $0 < \alpha < 1$, the functions f_U, f_V and f_W are all increasing. Note that we can not analyze the functions f_V and f_W for $\alpha > 1$, since they are only valid for $0 < \alpha < 1$. In Figure 1 we see the behavior of these functions for the values $\alpha = 0.2, \alpha = 0.5$ and $\alpha = 0.8$.

The fact that these probability density functions are increasing indicates that, in the weighted averages analyzed, the greatest weight is in the historical values closer to t (when $u \simeq 1$). Then, all the values $s \le t$ contribute to define $J_t^\alpha f(t), D_t^\alpha f(t)$ and $cD_t^\alpha f(t)$, especially the most recent ones. This is according to what is expected of an evolutionary epidemiological system.

Although recent values always have a greater weight than remote values, we see in the graph of f_U that this difference becomes smaller when α approaches 1. That is, the greater the value of $\alpha \in (0, 1)$, the more uniform is the contribution of historical values. On the other hand, in the graphics of f_V and f_W the opposite happens: the contribution becomes more uniform the smaller the value of $\alpha \in (0, 1)$ is. In fact, we see that the density function f_U, for the parameter α, coincides with f_V for the parameter $1-\alpha$.

Next, in order to illustrate the effectiveness of a fractional system and the memory effect, we present an application with an epidemiological model, using the Caputo derivative, to model the propagation of COVID-19 in Brazil.

4 A model with memory to study COVID-19 in Brazil

In order to testify the importance of the memory effect in modeling real phenomena, in this section we study the curve of active cases of COVID-19 from Brazil using the epidemiological SIR model, in its classic version and in the fractional version (with memory).

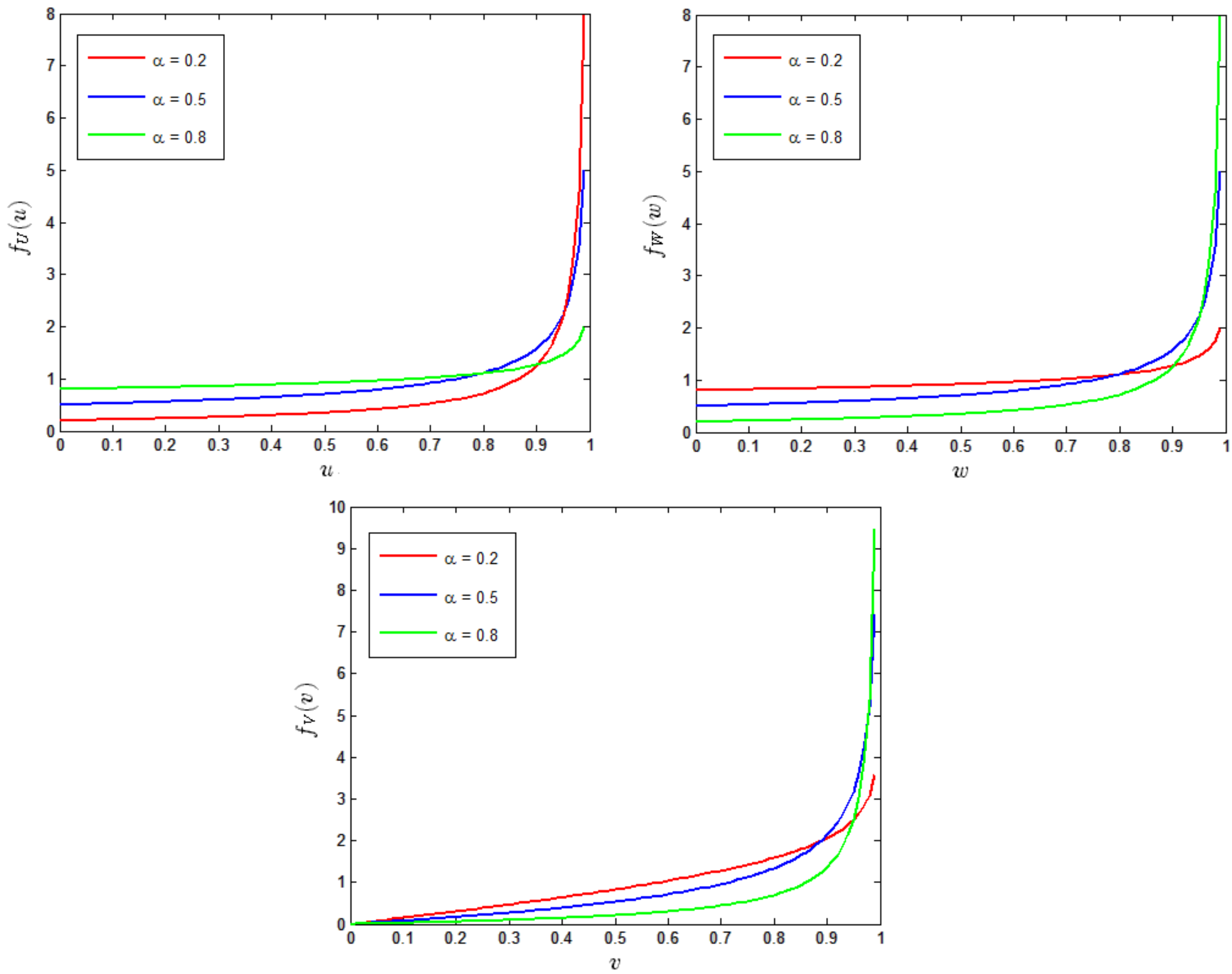

Figure 1: Density functions f_U, f_W and f_V, for parameters $\alpha = 0.2, \alpha = 0.5$ and $\alpha = 0.8$. a) shows the distribution of historical weights for fractional integral; b) illustrates the distribution of historical weights for the Caputo and Riemann-Liouville derivatives; c) shows the distribution that, combined with f_W, defines the historical weights for the Riemann-Liouville derivative.

The SIR model was proposed by Kermack and McKendrick in 1927 (KERMACK; MCKENDRICK, 1927) and presents the following formulation:

$$\begin{cases} S'(t) &= -\beta S(t)I(t) \\ I'(t) &= \beta S(t)I(t) - \gamma I(t) \\ R'(t) &= \gamma I(t) \end{cases}, \tag{18}$$

where $S(t)$, $I(t)$ and $R(t)$ are the number of susceptible, infected and recovered individuals at instant t, respectively. The parameter $\beta > 0$ is the transmission rate and $\gamma > 0$ is the recovery rate (EDELSTEIN-KESHET, 2005; KERMACK; MCKENDRICK, 1927).

In this paper, we used the fractional versions of the SIR model, given by (DIETHELM, 2013), as follows:

$$\begin{cases} cD_t^\alpha S(t) &= -\beta^\alpha S(t)I(t) \\ cD_t^\alpha I(t) &= \beta^\alpha S(t)I(t) - \gamma^\alpha I(t) \\ cD_t^\alpha R(t) &= \gamma^\alpha I(t) \end{cases}, \tag{19}$$

where cD_t^α is the fractional derivative of Caputo of order $\alpha \in (0,1)$.

In order to compare (18) and (19), we performed data fit using the least squares method, with data from active cases of COVID-19 in Brazil, obtained in (WORLDOMETERS WEBSITE, 2021). The initial condition is given by $R_0 = 0$, I_0 being the first value observed in the data and $S_0 = 1 - I_0$, since we consider $S, I, R \in [0,1]$.

For data normalization, we divided their values by N, the total number of individuals exposed to the disease. Note that, due to the control measures adopted by the countries, part of the population is not exposed to the propagation of the coronavirus. However, this is not considered in the SIR model. Then, if we consider N the total number of inhabitants in the country, the SIR model is not effective to fit the COVID-19 data. Therefore, we look for a new (smaller) value for this parameter to obtain good results in the fit data process. Thus, N becomes the number of individuals involved in the dynamic.

In addition to looking for the best data fit result, to determine the value of N we also consider another important factor: the gamma value being in the range $[0.07, 0.14]$. In the SIR model we know that $\frac{1}{\gamma}$ indicates the average time that a person remains infected. For COVID-19 we know that this time varies from 7-14 days on average and, therefore, we consider that $\gamma \in [0.07, 0.14]$.

Figure 2 shows the results obtained for the first "wave" of Brazil, where the blue dotted line represents the actual data, the red dashed line represents the fractional model solution (ie, with memory), and the black solid line represents the classic model solution. The value obtained for the number of individuals exposed to the disease is $N = 6 \cdot 10^6$.

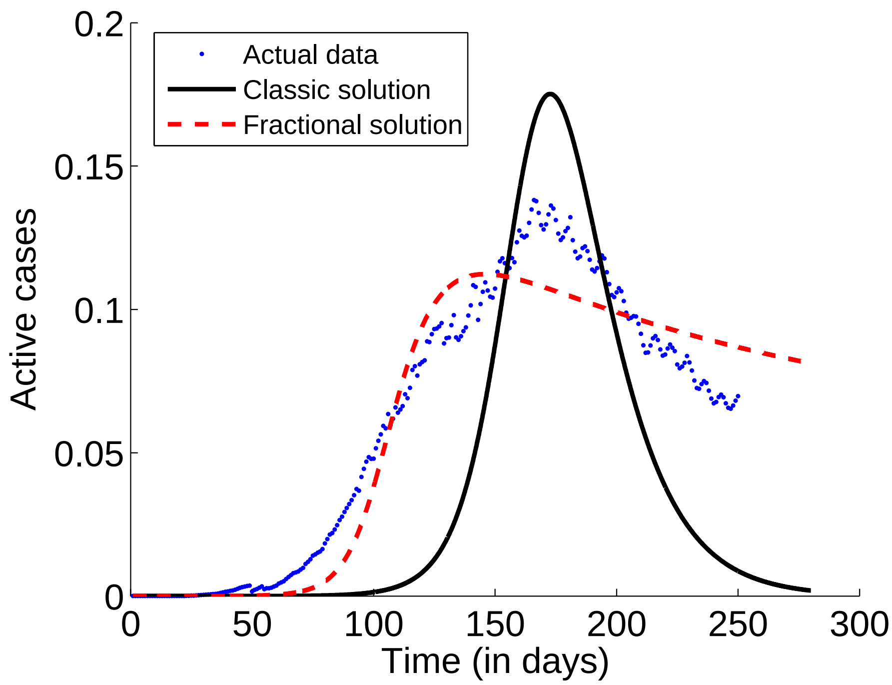

Figure 2: Solutions of the (18) and (19) to describe the spread of COVID-19 in the first "wave" of Brazil. For the model (18), $\beta = 0.1718$ and $\gamma = 0.0808$. For the model (19), $\beta = 0.5341$, $\gamma = 0.1045$ and $\alpha = 0.4458$.

We see that the model (19) presents the best data fit in the first "wave" and we can confirm this in Table 1, which shows the mean squared error value obtained by each model. Furthermore, using the model (19) we can analyze the degree of memory effect in the country, in the period referring to the first "wave", using the parameter α. We have seen that the smaller the value of α, the greater the influence of all historical values of I' ($I'(s), \forall s \leq t$) to determine the current variation given by $cD_t^\alpha I(t)$.

For the second "wave" in Brazil the models (18) and (19) are equally effective, the fractional model being little better, as we see in the Table 1. We also prove this fact in Figure 3, where the blue dotted line represents the actual data, the red dashed line represents the fractional model solution (ie, with memory), and the black solid line represents the classic model solution. The value obtained for the number of individuals exposed to the disease is $N = 250 \cdot 10^6$.

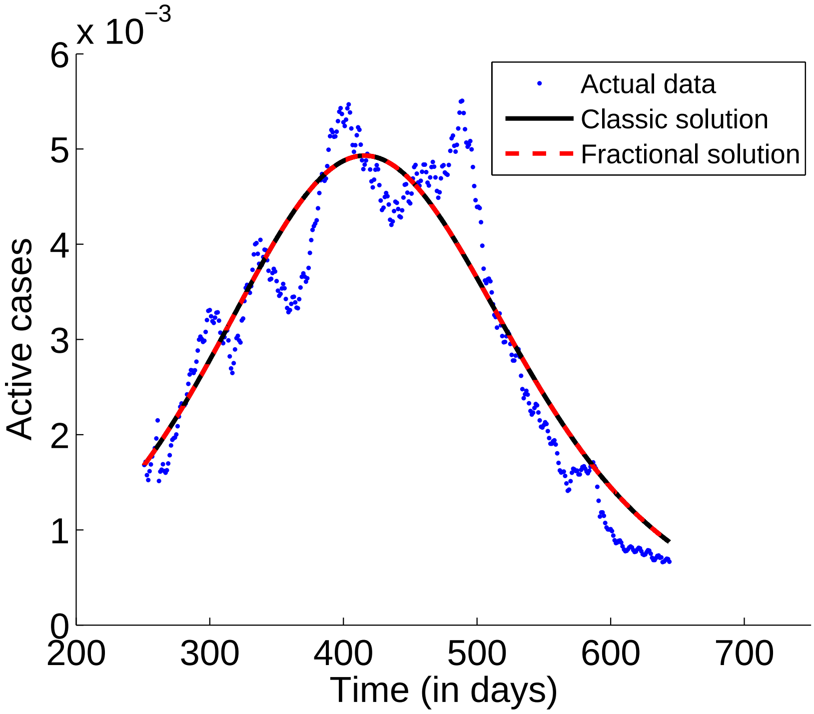

Figure 3: Solutions of the (18) and (19) to describe the spread of COVID-19 in the second "wave" of Brazil. For the model (18), $\beta = 0.1435$ and $\gamma = 0.129$. For the model (19), $\beta = 0.1427$, $\gamma = 0.130$ and $\alpha = 0.9999$.

	Classic SIR model	Fractional SIR model
First "wave"	0.3732	0.0376
Second "wave"	$8.3957 \cdot 10^{-5}$	$8.3945 \cdot 10^{-5}$

Table 1: Mean squared error values.

5 Conclusion

In this paper we propose a new approach to manage the memory effect in fractional calculus. More specifically, we check the memory effect on the Caputo derivative and integral and Riemann-Liouville derivative of a function f. We show that these operators are proportional to a weighted average of f and/or f'. Also, we analyzed the contribution of historical values and concluded that to determine the operators at the current moment, recent values have greater weight than remote values.

Moreover, the formulas we propose here allow the exploration of new concepts and interesting interpretations. For example, through them it is easy to see that when the fractional derivative vanishes at a point t^*, then the local maximum/minimum point occurs at $t < t^*$.

Finally, we compare a model with fractional differential equations, using Caputo derivative, with a classic model of ordinary differential equations, and show that the fractional model allows advantages such as better fitting the data and analyzing the degree of memory effect present in the dynamics. Furthermore, this model well described the propagation of COVID-19 in Brazil.

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