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Towards an algebrization of a linear temporal logic

Em direção à algebrização de uma lógica temporal linear

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Abstract

The subject of this paper is the Propositional Neighbourhood Logic (PNL), a case of linear temporal logic and an extension of Classical Propositional Logic that results from the inclusion of modal operators that deal with certain temporal aspects. In this context, the validity of a proposition is strictly related to some temporal interval. We highlight some formal aspects of this logic and its formalization in some deductive systems (or proof systems), which naturally reflect the formal aspects of time. We aim to investigate algebraic models for Propositional Neighbourhood Logic. However, in this work, we focus exclusively on presenting the Soundness result, one of the necessary steps towards obtaining an adequate (sound and complete) model, between the axiomatic system of PNL and the planned algebraic model.

Keywords: logic; algebraic logic; temporal logic; neighbourhood logic.

Resumo

O presente trabalho trata da Lógica Proposicional da Vizinhança (LPV), um caso de lógica temporal linear e uma extensão da Lógica Proposicional Clássica dada pelo acréscimo de operadores modais que tratam de certos aspectos temporais. Nesse contexto, a validade de uma proposição está estritamente relacionada a algum intervalo de tempo. Destacamos alguns aspectos formais dessa lógica e a sua formalização em sistemas dedutivos (ou de provas), que naturalmente retratam os aspectos formais do tempo. Pretendemos investigar modelos algébricos para a Lógica Proposicional da Vizinhança. Todavia, neste trabalho, nos dedicamos apenas à apresentação do resultado de Correção, uma das etapas necessárias para a obtenção de um modelo adequado (correto e completo), entre o sistema axiomático da LPV e o pretendido modelo algébrico.

Palavras-chave: lógica; lógica algébrica; lógica temporal; lógica da vizinhança.

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1 Introduction

The initial motivation for the development of this work is to address a formalization about the concept of time in a logical context, rich enough to involve aspects of mathematical analysis or real analysis.

For this, we looked for a temporal logic that was, in somewhat, contemporary, since there are many logics of time or temporal logics in the literature, but some quite basic. Besides that, we also looked in literature for some steps without plenty justifications, so that we could make some original contributions.

The logic found, which has some of the mentioned elements, was Neighbourhood Logic (NL), with emphasis on its propositional and modal part, which we began to investigate carefully and in some depth.

Fortunately, it can be seen that there are few texts on Neighbourhood Logic, as it is recent in someway, and that there is no well-founded and adequate algebraic model for this logic.

But, we can say that some authors, like Höfner and Möller, has paved part of the way. In Höfner (2007) and Höfner and Möller (2008), the authors develop what they call an algebraic embedding of Neighbourhood Logic. They show a kind of algebraic structure that handle an algebraic version of some NL-theorems. However, there is no attempt to establish a model properly, with Soundness and Completeness theorems. Although the papers provide a good intuition in how an algebraic model of this logic could be.

So, this is the goal established for the present work, evaluate a formalization of time, via a logic which we can deal with and contribute in some original aspect, the establishment of an algebraic model for the propositional and modal part of NL, the Propositional Neighbourhood Logic (PNL).

NL was introduced as a first-order logic, but we do not consider the quantificational part, since the interest was exactly in the algebraic behavior behind modal operators.

Moreover, this work extends Freitas, Feitosa and Silvestrini (2024) by the inclusion of some additional aspects and remarks, in addition the proof of main results of the work. Finally, as a remark, the present paper deal only with the Soundness Theorem, the main subject of Freitas, Feitosa and Silvestrini (2024), and we will let the Completeness for a next work.

2 Propositional Neighbourhood Logic

This section will only be responsible for the presentation of PNL axiomatic system, as it is a necessary part for the development of the Soundness result. However, we only show general and essential notions.

In case the reader is interested, we recommend reading Goranko, Montanari and Sciavicco (2003); a work in which the authors develop some axiomatic systems for what they name Propositional Neighbourhood Logics (PNL's).

For these authors, PNL's are a family of interval temporal logics (propositional and modal only). This family of temporal logics also includes *non-strict propositional neighbourhood logics* and *strict propositional neighbourhood logics*. What for the Goranko, Montanari and Sciavicco (2003) is a non-strict propositional neighbourhood logic, for us is the Propositional Neighbourhood Logic, the propositional part of Chaochen and Hansen's (1997, 2004) Neighbourhood Logic.

In the words of Goranko, Montanari and Sciavicco,

> unlike classical logic and most modal and temporal logics, where the first-order axiomatic

> systems are obtained by extending their propositional fragments with relevant axioms for the quantifiers, the first-order NL was axiomatized first, without its propositional fragment having been identified. It now turns out that the latter was hidden into the originally introduced first-order axiomatic system, the propositional axioms of which, taken alone, are substantially incomplete. (Goranko; Montanari; Sciavicco, 2003, p. 1140).

For now, we are going to show the axiomatic system which is said to be Sound and Complete in according to a relational semantic for PNL (Goranko; Montanari; Sciavicco, 2003, p. 1152).

2.1 Axiomatic system

This formal system will be indicated by $\mathcal{L}_{PNL}$.

The language L of the $\mathcal{L}_{PNL}$ is defined as usual over the following set of propositional symbols:

$$L = \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \Box_l, \Diamond_l, \Box_r, \Diamond_r, p_1, p_2, p_3, \ldots, p_n\}.$$

The following notations are going to be used:

$Var(\mathcal{L}_{PNL})$ is the set of propositional variables of the system. And so, $Var(\mathcal{L}_{PNL}) = \{p_1, p_2, p_3, \ldots, p_n\}$;

$For(\mathcal{L}_{PNL})$ denotes the set of formulas of the system defined in an inductively way:

- if $i \in \mathbb{N}$ and $p_i \in Var(\mathcal{L}_{PNL})$, then $p_i \in For(\mathcal{L}_{PNL})$;
- if $\varphi, \psi \in For(\mathcal{L}_{PNL})$, then $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$, $\varphi \leftrightarrow \psi \in For(\mathcal{L}_{PNL})$;
- if $\varphi \in For(\mathcal{L}_{PNL})$, then $\Box_*\varphi$ and $\Diamond_*\varphi \in For(\mathcal{L}_{PNL})$, for any $* \in \{l, r\}$;

Set of formulas are denoted by Γ or Δ.

The symbol $\vdash$ is used as a syntactical consequence or deductive consequence.

If $\Gamma \vdash \varphi$ is the case, then the formula φ is a deductive consequence of the set of formulas Γ.

Lastly, Γ can be an empty set and when this is the case, then we say that φ is a theorem of $\mathcal{L}_{PNL}$ and just write $\vdash \varphi$.

Scheme of Axioms

(Ax0) CPL

(Ax1) $\Box_*(\varphi \rightarrow \psi) \rightarrow (\Box_*\varphi \rightarrow \Box_*\psi)$

(Ax2) $\Box_*\varphi \rightarrow \Diamond_*\varphi$

(Ax3) $\varphi \rightarrow \Box_r\Diamond_l\varphi$ and $\varphi \rightarrow \Box_l\Diamond_r\varphi$

(Ax4) $\Diamond_r\Diamond_l\varphi \rightarrow \Box_r\Diamond_l\varphi$ and $\Diamond_l\Diamond_r\varphi \rightarrow \Box_l\Diamond_r\varphi$

(Ax5) $\Diamond_*\Diamond_*\Diamond_*\varphi \rightarrow \Diamond_*\Diamond_*\varphi$

(Ax6) $\Box_r\Diamond_l\varphi \rightarrow \Diamond_l\Diamond_r\Diamond_r\varphi \vee \Diamond_l\Diamond_l\Diamond_r\varphi$ and $\Box_l\Diamond_r\varphi \rightarrow \Diamond_r\Diamond_l\Diamond_l\varphi \vee \Diamond_r\Diamond_r\Diamond_l\varphi$

(Ax7) $\Box_*\varphi \wedge \Diamond_*\psi_1 \wedge \ldots \wedge \Diamond_*\psi_n \rightarrow \Diamond_*(\Box_*\varphi \wedge \Diamond_*\psi_1 \wedge \ldots \wedge \Diamond_*\psi_n)$, for any $1 \leq n$.

Inference Rules

(MP) If $\varphi \in \Delta$ and $\varphi \rightarrow \psi \in \Delta$, then $\Delta \vdash \psi$ by the rule (MP)

(RN) If $\vdash \varphi$, then $\vdash \Box_*\varphi$.

The axiom (Ax0) says that every classical propositional theorem is a theorem of $\mathcal{L}_{PNL}$.

The modal operators marked with $*$ indicate one of the sub-indices 'r' or 'l', and in this case we understand that the modal formula is valid for any 'r' or 'l'.

There are some inter-definitions between the modal operators $\Box$ and $\Diamond$ like:

$$\Box_* \varphi = \neg \Diamond_* \neg \varphi \qquad\qquad \Diamond_* \varphi = \neg \Box_* \neg \varphi.$$

A meaning and intuition for the modal operators can be gathered from PNL relational/Kripke semantics in (Goranko; Montanari; Sciavicco, 2003), but also from Neighbourhood Logic semantics in Chaochen and Hansen (1997, 2004).

For $\Diamond_l \varphi$ case, we must read as "exists a non-strict interval concatened on the left (on the past) which satisfies its sub-formula φ". While $\Diamond_r \varphi$ says the same but thinking on the right (on the future) instead.

On the other hand, $\Box_*$ operators says that "every non-strict interval concatened satisfies its sub-formula". If we choose $* = l$, then we are talking about past (on the left) intervals. But if we take $* = r$, then we are talking about future (on the right) intervals.

3 Some algebraic elements

In this section, we present all the algebraic machinery necessary.

We introduce the algebraic structures and part of the properties that will compose the algebraic model for the Propositional Neighbourhood Logic. And in a more complete way than we have done in Freitas, Feitosa and Silvestrini (2024), we will include details and proofs.

For the reader to understand some definitions and results in this section, they must be familiar with topics on lattices and Boolean algebras. As we take these topics as elementary, we only refer to some canonical readings as Birkhoff (1948) and Rasiowa (1974).

Definition 3.1 *Monoid is a terna $\langle M, *, \varepsilon \rangle$, such that M is a non-empty set, $*$ is a binary operation in M, ε is a constant of M, and:*

(i) *the operation $*$ is associative;*

(ii) *the constant ε is a neutral element for the operation $*$.*

Definition 3.2 *Semiring is a structure $\langle S, +, \circ, 0, 1' \rangle$ such that S is a non-empty set, $+, \circ$ are binary operations over S, $0, 1'$ are constants of S, and for every $x, y, z \in S$:*

(i) *$\langle S, +, 0 \rangle$ is a commutative monoid;*

(ii) *$\langle S, \circ, 1' \rangle$ is a monoid;*

(iii) *$x \circ (y + z) = (x \circ y) + (x \circ z)$ and $(x + y) \circ z = (x \circ z) + (y \circ z)$;*

(iv) *$x \circ 0 = 0 = 0 \circ x$.*

The element 0 is an absorbing element for the operation $\circ$. In general, we call the operation $+$ of addition and the operation $\circ$ of composition.

Definition 3.3 *For any $x \in S$, the structure $\langle S, +, \circ, 0, 1' \rangle$ is an idempotent semiring if:*

(i) *$\langle S, +, \circ, 0, 1' \rangle$ is a semiring;*

(ii) *$x + x = x$.*

The idempotence is characterized exactly for the addition operation $+$.

In idempotent semirings, the ordering relation can be seen in this way:

$$x \leq y \Leftrightarrow x + y = y.$$

The validity of idempotence law is necessary in order to hold the reflexivity property of the order relation, only obtained by the idempotence.

Proposition 3.4 *For any idempotent semiring* $\langle S, +, \circ, 0, 1' \rangle$*:*

(i) $\leq$ *is a partial order relation;*

(ii) 0 *is the least element by the ordering* $\leq$.

Proof: (i) *By idempotence* $x + x = x$ *and so* $x \leq x$*, the reflexivity holds.*

If $x \leq y$ *and* $y \leq z$*, then* $x + y = y$ *and* $y + z = z$*. It follows that* $x + z = x + (y + z) = (x + y) + z = y + z = z$*, and so* $x \leq z$*, thus we have the transitivity.*

If $x \leq y$ *and* $y \leq x$*, then* $x + y = y$ *and* $x + y = x$*. It follows that* $x = x + y = y$ *and we have anti-symmetry.*

Since for $\leq$ *the reflexivity, transitivity and anti-symmetry holds, we can conclude that it is a partial order relation.*

(ii) *As* 0 *is neutral for* $+$*, then for any* $x \in S$*, it follows that* $0 + x = x$ *and so* $0 \leq x$*, for all* $x \in S$*.*

■

In a semiring, 0 is the least element, but $1'$ is not the greatest.

Proposition 3.5 *If* $\langle S, +, \circ, 0, 1' \rangle$ *is an idempotent semiring, then:*

(i) $x \leq y \Rightarrow x + z \leq y + z$*;*

(ii) $x \leq y \Rightarrow z + x \leq z + y$*;*

(iii) $x \leq y \Rightarrow x \circ z \leq y \circ z$*;*

(iv) $x \leq y \Rightarrow z \circ x \leq z \circ y$*;*

(v) *if* $x \leq y$ *and* $w \leq z$*, then* $x \circ w \leq y \circ z$*;*

(vi) *if* $x \leq y$ *and* $w \leq z$*, then* $w \circ x \leq z \circ y$*.*

Proof: (i) $x \leq y \Rightarrow x + y = y \Rightarrow (x + y) + z = y + z \Rightarrow (x + y) + (z + z) = y + z \Rightarrow (x + z) + (y + z) = y + z \Rightarrow x + z \leq y + z$.

(ii) $x \leq y \Rightarrow x + y = y \Rightarrow z + (x + y) = z + y \Rightarrow (z + z) + (x + y) = z + y \Rightarrow (z + x) + (z + y) = z + y \Rightarrow z + x \leq z + y$.

(iii) $x \leq y \Rightarrow x + y = y \Rightarrow (x + y) \circ z = y \circ z \Rightarrow (x \circ z) + (y \circ z) = y \circ z \Rightarrow x \circ z \leq y \circ z$.

(iv) $x \leq y \Rightarrow x + y = y \Rightarrow z \circ (x + y) = z \circ y \Rightarrow (z \circ x) + (z \circ y) = z \circ y \Rightarrow z \circ x \leq z \circ y$.

(v) *Since* $x \leq y$ *and* $w \leq z$*, then* $x \circ w \leq y \circ w$ *and* $y \circ w \leq y \circ z$*. By transitivity of the order relation, it follows that* $x \circ w \leq y \circ z$*.*

(vi) *Since* $x \leq y$ *and* $w \leq z$*, then* $w \circ x \leq w \circ y$ *and* $w \circ y \leq z \circ y$*. By transitivity of the order relation, it follows that* $w \circ x \leq z \circ y$*.* ■

The $+$ operation does not yet characterize the concept of *supremum*, for the order $\leq$, since in an idempotent semiring the absorption laws do not hold in general. These laws are necessary to demonstrate that, for any $x, y \in S$, $x, y \leq x + y$ is the case, in other words, that $x + y$ is an upper bound of $\{x, y\}$.

For the purpose of get the greatest element for this algebraic structure, we will need some other operations and elements that define a Boolean semiring.

We take $\langle S, +, \cdot, \circ, \sim, 0, 1', 1 \rangle$, which extends the semiring by the inclusion of $\sim$ unary operation, $\cdot$ binary operation, and the constant $1 \in S$.

Definition 3.6 *The structure* $\langle S, +, \cdot, \circ, \sim, 0, 1', 1 \rangle$ *is a Boolean semiring if:*

(i) $\langle S, +, \circ, 0, 1' \rangle$ *is a semiring;*

(ii) $\langle S, +, \cdot, \sim, 0, 1 \rangle$ *is a Boolean algebra (Boolean lattice).*

Thus, the structure $\langle S, +, \cdot, \sim, 0, 1\rangle$ has the element 1 as the greatest element, and the operation $\sim$ as the Boolean complement. Since the underlying lattice of the semiring $\langle S, +, \cdot, \sim, 0, 1\rangle$ is Boolean, then $+$ must be seen as a *supremum* and $\cdot$ as an *infimum* in S.

Definition 3.7 *In an idempotent semiring $\langle S, +, \circ, 0, 1'\rangle$, subidentity is an element $a \in S$ such that $a \leq 1'$.*

To define the elements considered as subidentities, it is sufficient that the structure be - at least - an idempotent semiring, given that the ordering relation is necessary.

In the next steps, we will consider the Boolean semiring:

$$\mathbf{S} = \langle S, +, \cdot, \circ, \sim, 0, 1', 1\rangle.$$

Let $\mathfrak{S}(S)$ be the set of all subidentities of the semiring S. Thus, $\mathfrak{S}(S) = \{a : a \in S \text{ and } a \leq 1'\}$.

Definition 3.8 *In $\mathfrak{S}(S)$, the complement of $a \in \mathfrak{S}(S)$ is given by:*

$$-a := \sim a \cdot 1'.$$

Below, we present some properties about subidentities.

Proposition 3.9 *Let $\mathbf{S}$ a Boolean semiring and $a, b \in \mathfrak{S}(S) \subseteq S$. The following assertions are valid:*

(i) $0, 1' \in \mathfrak{S}(S)$;

(ii) *the set $\mathfrak{S}(S)$ is closed under the operations $+$, $\cdot$ and $\circ$;*

(iii) $a \circ b \leq a \cdot b$;

(iv) $a \cdot (-a) = 0$ *and* $a + (-a) = 1'$;

(v) *the operation $\circ$ is idempotent in $\mathfrak{S}(S)$;*

(vi) $a \circ b = a \cdot b$, *for all* $a, b \in \mathfrak{S}(S)$.

Proof: (i) *Since the semiring is Boolean, so 0 is the least element and $0 \leq 1'$, thus $0 \in \mathfrak{S}(S)$. By reflexivity, it follows that $1' \leq 1'$ and so $1' \in \mathfrak{S}(S)$.*

(ii) *If $a, b \in \mathfrak{S}(S) \subseteq S$, then $a \leq 1'$ and $b \leq 1'$.*

Since the operations $+$ and $\cdot$ are sup and inf, then $a + b \leq 1'$ and so $a + b \in \mathfrak{S}(S)$. On the other hand, $a \cdot b \leq 1' \cdot 1' = 1'$, thus $a \cdot b \in \mathfrak{S}(S)$.

In the case of operation $\circ$, $a \leq 1' \Rightarrow a \circ b \leq 1' \circ b$. As $1'$ is a neutral element for the operation and $b \leq 1'$, it follows that $a \circ b \leq b \leq 1'$. Thus, $a \circ b \leq 1'$.

Hence, we can conclude that $\mathfrak{S}(S)$ is closed under $+$, $\cdot$ and $\circ$.

(iii) $a \leq 1' \Rightarrow a \circ b \leq 1' \circ b \Rightarrow a \circ b \leq b$ *e* $b \leq 1' \Rightarrow a \circ b \leq a \circ 1' \Rightarrow a \circ b \leq a$. *Whence it follows that $a \circ b \leq a \cdot b$.*

(iv) $a \cdot (-a) = a \cdot (\sim a \cdot 1') = (a \cdot (\sim a)) \cdot 1' = 0 \cdot 1' = 0$.

$a + (-a) = a + (\sim a \cdot 1') = (a + (\sim a)) \cdot (a + 1') = 1 \cdot (a + 1') = a + 1'$. *Since $+$ is the sup and $1'$ is the greatest element of $\mathfrak{S}(S)$, so the greatest element between a and $1'$ is $1'$ and, thus, $a + 1' = 1'$.*

(v) *As $a \leq 1'$, then $a \circ a \leq 1' \circ a = a$.*

On the other hand, $a = 1' \circ a = (a + (-a)) \circ a = (a \circ a) + (a \circ (-a))$. Since $a \circ a \leq a \circ a$ and $a \circ (-a) \leq a \cdot (-a)$ by the item (iii)*, it follow that $(a \circ a) + (a \circ (-a)) \leq (a \circ a) + (a \cdot (-a))$ and so $(a \circ a) + (a \circ (-a)) \leq (a \circ a) + 0 \Rightarrow (a \circ a) + (a \circ (-a)) \leq (a \circ a) \Rightarrow a \leq a \circ a$. Hence, $a \circ a = a$.*

(vi) *In order to demonstrate that $a \circ b = a \cdot b$, we are going to show that $a \cdot b \leq a \circ b$, since we already demonstrated $a \circ b \leq a \cdot b$ in the item* (iii).

It follows that $a \cdot b \leq a$ and $a \cdot b \leq b$ implies $(a \cdot b) \circ (a \cdot b) \leq a \circ b$. As idempotence holds for $\circ$, we can conclude that $a \cdot b \leq a \circ b$. ■

By the above proposition, the set $\mathfrak{S}(S)$ is closed under the operations $+$, $\cdot$ and $\circ$, has 0 and $1'$ as least and greatest element (zero and unity), respectively, and there is defined an operation of complement in it. And, not only is defined a complement in $\mathfrak{S}(S)$, but also every $a \in \mathfrak{S}(S)$ has a complement in $\mathfrak{S}(S)$: we just need to see that by definition $-a = \sim a \cdot 1' \leq 1'$ and so $-a \in \mathfrak{S}(S)$. Thus, as we can see, $\mathfrak{S}(S)$ forms a complemented distributive lattice with zero and unity, i. e., $\mathfrak{S}(S)$ constitutes a Boolean algebra.

It is interesting to note that the operations $\cdot$ and $\circ$ collapse in $\mathfrak{S}(S)$, that is, for $a, b \in \mathfrak{S}(S)$, we have $a \cdot b = a \circ b$.

Observation 3.10 *If the structure $\langle S, \mathfrak{S}(S), +, \circ, \cdot, \sim, -, 0, 1', 1\rangle$ is a Boolean semiring with subidentities, with $\mathfrak{S}(S) \subseteq S$, then:*

(i) $\langle S, +, \circ, \cdot, \sim, 0, 1', 1\rangle$ *is a Boolean semiring;*

(ii) $\langle \mathfrak{S}(S), +, \circ, -, 0, 1'\rangle$ *is a Boolean algebra.*

For economy, this structure is going to be evoked only by $\langle S, \mathfrak{S}(S)\rangle$, such that **S** is the Boolean semiring and $\mathfrak{S}(S)$ a Boolean algebra constituted by the subidentities of **S**.

Proposition 3.11 *In $\langle S, \mathfrak{S}(S)\rangle$ the following statements are valid for any $a, b \in \mathfrak{S}(S)$ and $x \in S$:*

(i) $(a \cdot b) \circ x = (a \circ x) \cdot (b \circ x)$ *and* $x \circ (a \cdot b) = (x \circ a) \cdot (x \circ b)$*;*

(ii) $\sim (a \circ 1) = -a \circ 1$ *and* $\sim (1 \circ a) = 1 \circ (-a)$*;*

(iii) $x \circ a \leq x$*;*

(iv) $a \circ x \leq x$*;*

(v) $x \leq x \circ 1$*;*

(vi) $x \leq 1 \circ x$*;*

(vii) $1 \circ 1 = 1$.

Proof: (i) *Let's see that $a \cdot b \leq a \Rightarrow (a \cdot b) \circ x \leq a \circ x$ and $a \cdot b \leq b \Rightarrow (a \cdot b) \circ x \leq b \circ x$. Thus, $(a \cdot b) \circ x \leq (a \circ x) \cdot (b \circ x)$. In an analogous way, it turns out that $a \cdot b \leq a \Rightarrow x \circ (a \cdot b) \leq x \circ a$ and $a \cdot b \leq b \Rightarrow x \circ (a \cdot b) \leq x \circ b$. Hence, $x \circ (a \cdot b) \leq (x \circ a) \cdot (x \circ b)$.*

For this side of the inequality we don't need the fact that a and b are subidentities, but for the other side we need the condition.

On the other side, $(a \circ x) \cdot (b \circ x) = 1' \circ ((a \circ x) \cdot (b \circ x)) = (a + (-a)) \circ ((a \circ x) \cdot (b \circ x)) = (a \circ ((a \circ x) \cdot (b \circ x))) + (-a \circ ((a \circ x) \cdot (b \circ x)))$. As $(a \circ x) \cdot (b \circ x) \leq b \circ x \Rightarrow a \circ ((a \circ x) \cdot (b \circ x)) \leq a \circ (b \circ x)$ and $(a \circ x) \cdot (b \circ x) \leq a \circ x \Rightarrow -a \circ ((a \circ x) \cdot (b \circ x)) \leq -a \circ (a \circ x)$, it follows that $(a \circ ((a \circ x) \cdot (b \circ x))) + (-a \circ ((a \circ x) \cdot (b \circ x))) \leq (a \circ (b \circ x)) + (-a \circ (a \circ x))$. Finally, $(a \circ (b \circ x)) + (-a \circ (a \circ x)) = ((a \circ b) \circ x) + ((-a \circ a) \circ x) = ((a \cdot b) \circ x) + (0 \circ x) = ((a \cdot b) \circ x) + 0 = (a \cdot b) \circ x$ and so $(a \circ x) \cdot (b \circ x) \leq (a \cdot b) \circ x$.

For the other case, $(x \circ a) \cdot (x \circ b) = ((x \circ a) \cdot (x \circ b)) \circ 1' = ((x \circ a) \cdot (x \circ b)) \circ (b + (-b)) = (((x \circ a) \cdot (x \circ b)) \circ b) + (((x \circ a) \cdot (x \circ b)) \circ -b)$. In a similar way to the paragraph above, $(((x \circ a) \cdot (x \circ b)) \circ b) + (((x \circ a) \cdot (x \circ b)) \circ -b) \leq ((x \circ a) \circ b) + ((x \circ b) \circ -b)$. And since $((x \circ a) \circ b) + ((x \circ b) \circ -b) = (x \circ (a \circ b)) + (x \circ (b \circ -b)) = (x \circ (a \cdot b)) + (x \circ 0) = (x \circ (a \cdot b)) + 0 = x \circ (a \cdot b)$, then $(x \circ a) \cdot (x \circ b) \leq x \circ (a \cdot b)$.

(ii) *This proof comes from two different paths, it will be seen that $(a \circ 1) + (-a \circ 1) = 1$ and $(a \circ 1) \cdot (-a \circ 1) = 0$ and, hence, that $(-a \circ 1)$ is the complement of $(a \circ 1)$, or even,* *that* $\sim (a \circ 1) = (-a \circ 1)$. *Thereby:* $(a \circ 1) + (-a \circ 1) = (a + (-a)) \circ 1 = 1' \circ 1 = 1$ *and* $(a \circ 1) \cdot (-a \circ 1) = (a \cdot (-a)) \circ 1 = 0 \circ 1 = 0$

Analogously, it turns out that $(1 \circ a) + (1 \circ -a) = 1 \circ (a + (-a)) = 1 \circ 1' = 1$ *and* $(1 \circ a) \cdot (1 \circ -a) = 1 \circ (a \cdot (-a)) = 1 \circ 0 = 0$. *Thus, we conclude that* $(1 \circ -a)$ *is the complement of* $(1 \circ a)$, *i. e.,* $\sim (1 \circ a) = (1 \circ -a)$.

(iii) *Since* $a \in \mathfrak{S}(S)$, *it follows that* $a \leq 1' \Rightarrow x \circ a \leq x \circ 1' \Rightarrow x \circ a \leq x$.

(iv) *In the same way, since* $a \in \mathfrak{S}(S)$, *it follows that* $a \leq 1' \Rightarrow a \circ x \leq 1' \circ x \Rightarrow a \circ x \leq x$.

(v) $1' \leq 1 \Rightarrow x \circ 1' \leq x \circ 1$. *As* $1'$ *is the neutral element for composition, then* $x \leq x \circ 1$.

(vi) $1' \leq 1 \Rightarrow 1' \circ x \leq 1 \circ x$. *As* $1'$ *is the neutral element for composition, then* $x \leq 1 \circ x$.

(vii) *As* $1 \circ 1 \leq 1$. *On the other hand, using the previous items* $1 \leq 1 \circ 1$ *and so* $1 = 1 \circ 1$. ■

Lastly, we need to define the domain and codomain operators.

Definition 3.12 *The domain operator* $\ulcorner : S \to \mathfrak{S}(S)$ *is an unary operation which satisfies the following conditions:*

(i) $x \leq \ulcorner x \circ x$;

(ii) $\ulcorner(a \circ x) \leq a$.

The letters x, y, z will be used for the elements of S and a, b, c for the elements of $\mathfrak{S}(S)$.

It is important to remark that elements of S operated by operators $\ulcorner$ and its symmetric $\urcorner$ are elements of $\mathfrak{S}(S)$ as well. This is a direct consequence of items (ii) of the domain and codomain definitions, given that if we take $a = 1'$ we will have $\ulcorner(1' \circ x) \leq 1'$ and, therefore, $\ulcorner x \leq 1'$.

Desharnais, Möller and Struth (2004), as well as Höfner (2007), define the domain operator as in the definition above, and in the case of the operator also satisfying the property $\ulcorner(x \circ \ulcorner y) \leq \ulcorner(x \circ y)$, then the semiring is called a modal semiring.

However, there are slightly different approaches in the literature. Höfner and Möller (2008) and Möller (2004, 2007), for example, define the domain operator as an operator that satisfies the three properties, while Desharnais, Möller and Struth (2003) name by pre-domain if the operator satisfies only the first two properties, and domain if it also satisfies the third one.

The approach of the present work follows Desharnais, Möller and Struth (2004) and Höfner (2007) in some way.

Definition 3.13 *The domain operator* $\ulcorner : S \to \mathfrak{S}(S)$ *is named modal if, in addition to having properties of a domain operator, it also satisfies the following property:*

(iii) $\ulcorner(x \circ \ulcorner y) \leq \ulcorner(x \circ y)$.

Definition 3.14 *The codomain operator* $\urcorner : S \to \mathfrak{S}(S)$ *is an unary operation which satisfies the following conditions:*

(i) $x \leq x \circ x^{\urcorner}$;

(ii) $(x \circ a)^{\urcorner} \leq a$.

Definition 3.15 *The codomain operator* $\urcorner : S \to \mathfrak{S}(S)$ *is named modal if, in addition to having properties of a codomain operator, it also satisfies the following property:*

(iii) $(x^{\urcorner} \circ y)^{\urcorner} \leq (x \circ y)^{\urcorner}$.

By the union of these two operators with the Boolean semiring with subidentities, we obtain the following algebraic structures.

Definition 3.16 *A Boolean semiring with bidomain is a structure $\langle\langle S, \mathfrak{S}(S)\rangle, {}^{\ulcorner}, {}^{\urcorner}\rangle$, in which:*
(i) $\langle S, \mathfrak{S}(S)\rangle$ *is a Boolean semiring with subidentities;*
(ii) ${}^{\ulcorner}$ *and* ${}^{\urcorner}$ *are domain and codomain operators, respectively.*

Definition 3.17 *A Boolean semiring with modal bidomain is a structure $\langle\langle S, \mathfrak{S}(S)\rangle, {}^{\ulcorner}, {}^{\urcorner}\rangle$, in which:*
(i) $\langle S, \mathfrak{S}(S)\rangle$ *is a Boolean semiring with subidentities;*
(ii) ${}^{\ulcorner}$ *and* ${}^{\urcorner}$ *are modal domain and codomain operators, respectively.*

Proposition 3.18 *If $\langle\langle S, \mathfrak{S}(S)\rangle, {}^{\ulcorner}\rangle$ is an algebraic structure of semiring with subidentities, equipped with a domain operator, then the following statements are valid:*
(i) ${}^{\ulcorner}x \leq a \Rightarrow x \leq a \circ x$ *if, and only if,* $x \leq {}^{\ulcorner}x \circ x$*;*
(ii) $x \leq a \circ x \Rightarrow {}^{\ulcorner}x \leq a$ *if, and only if,* ${}^{\ulcorner}(a \circ x) \leq a$*.*
Proof: (i) $[\Rightarrow]$ *Since* ${}^{\ulcorner}x \leq {}^{\ulcorner}x$ *and, by the hypothesis,* ${}^{\ulcorner}x \leq {}^{\ulcorner}x \Rightarrow x \leq {}^{\ulcorner}x \circ x$*, then* $x \leq {}^{\ulcorner}x \circ x$*.*
$[\Leftarrow]$ *If* ${}^{\ulcorner}x \leq a$*, then* ${}^{\ulcorner}x \circ x \leq a \circ x$*. Since, by the hypothesis,* $x \leq {}^{\ulcorner}x \circ x$*, by the transitivity of* $\leq$*, we obtain* $x \leq a \circ x$*.*
(ii) $[\Rightarrow]$ *Certainly* $a \circ x \leq a \circ x$ *and, by the idempotence, it follows that* $a \circ x \leq (a \circ a) \circ x$ *and, by the associativity, that* $a \circ x \leq a \circ (a \circ x)$*. Using the hypothesis, we can conclude that* ${}^{\ulcorner}(a \circ x) \leq a$*.*
$[\Leftarrow]$ *As* $a \in \mathfrak{S}(S)$*, by the Definition 3.7,* $a \leq 1'$*. So,* $a \circ x \leq 1' \circ x$ *and since* $1'$ *is neutral for* $\circ$*, then* $a \circ x \leq x$*. Since* $x \leq a \circ x$*, thus* $x = a \circ x$*. By the hypothesis* ${}^{\ulcorner}(a \circ x) \leq a$ *and so* ${}^{\ulcorner}x \leq a$*.* ■

It follows from the above result that there is another way to define the operator of domain by the union of items (i) and (ii), generating the following Galois pair:

$$x \leq a \circ x \Leftrightarrow {}^{\ulcorner}x \leq a.$$

Symmetrically, this result is obtained for the codomain operator.

Proposition 3.19 *If $\langle\langle S, \mathfrak{S}(S)\rangle, {}^{\urcorner}\rangle$ is an algebraic structure of semiring with subidentities, equipped with a codomain operator, then the following statements are valid:*
(i) $x^{\urcorner} \leq a \Rightarrow x \leq x \circ a$ *if, and only if,* $x \leq x \circ x^{\urcorner}$*;*
(ii) $x \leq x \circ a \Rightarrow x^{\urcorner} \leq a$ *if, and only if,* $(x \circ a)^{\urcorner} \leq a$*.*
Proof: (i) $[\Rightarrow]$ *As* $x^{\urcorner} \leq x^{\urcorner}$*, then by the hypothesis* $x^{\urcorner} \leq x^{\urcorner} \Rightarrow x \leq x \circ x^{\urcorner}$ *and thus we have that* $x \leq x \circ x^{\urcorner}$*.*
$[\Leftarrow]$ *If* $x^{\urcorner} \leq a$*, then* $x \circ x^{\urcorner} \leq x \circ a$*. Since by the hypothesis* $x \leq x \circ x^{\urcorner}$*, with the transitivity of* $\leq$ *we obtain that* $x \leq x \circ a$*.*
(ii) $[\Rightarrow]$ *Considering that* $x \circ a \leq x \circ a$*, by the idempotence of* $\circ$*, it follows that* $x \circ a \leq x \circ (a \circ a)$ *and, by associativity,* $x \circ a \leq (x \circ a) \circ a$*. Using the hypothesis, it is concluded that* $(x \circ a)^{\urcorner} \leq a$*.*
$[\Leftarrow]$ *Since* $a \in \mathfrak{S}(S)$*, by the Definition 3.7,* $a \leq 1'$*. Thus,* $x \circ a \leq x \circ 1'$ *and as* $1'$ *is neutral for the operation* $\circ$*, then* $x \circ a \leq x$*. Given that* $x \leq x \circ a$*, so* $x = x \circ a$*. By the hypothesis* $(x \circ a)^{\urcorner} \leq a$ *and so* $x^{\urcorner} \leq a$*.* ■

As a consequence of the above result, the union of items (i) and (ii) culminate in a new property and possibility of defining the codomain operator:

$$x^{\urcorner} \leq a \Leftrightarrow x \leq x \circ a.$$

In the next step, we will demonstrate another property that can be useful in the course of developments.

Proposition 3.20 *If $\langle\langle S, \mathfrak{S}(S)\rangle, \ulcorner, \urcorner\rangle$ is a semiring with subidentities, equipped with domain and codomain operators, then the following statements are valid:*

(i) $\ulcorner x \leq a \Leftrightarrow -a \circ x \leq 0$;

(ii) $x^\urcorner \leq a \Leftrightarrow x \circ -a \leq 0$.

Proof: (i) *Given that $\ulcorner x \leq a \Leftrightarrow x \leq a \circ x$, as a result of Proposition 3.18, it remains to show that $x \leq a \circ x \Leftrightarrow -a \circ x \leq 0$.*

$[\Rightarrow]$ $x \leq a \circ x \Rightarrow -a \circ x \leq -a \circ (a \circ x) \Rightarrow -a \circ x \leq (-a \circ a) \circ x \Rightarrow -a \circ x \leq 0 \circ x \Rightarrow -a \circ x \leq 0$;

$[\Leftarrow]$ $-a \circ x \leq 0 \Rightarrow (a \circ x) + (-a \circ x) \leq (a \circ x) + 0 \Rightarrow (a + -a) \circ x \leq a \circ x \Rightarrow 1' \circ x \leq a \circ x \Rightarrow x \leq a \circ x$.

(ii) *Given that $x^\urcorner \leq a \Leftrightarrow x \leq x \circ a$, as a result of Proposition 3.19, it remains to show that $x \leq x \circ a \Leftrightarrow x \circ -a \leq 0$.*

$[\Rightarrow]$ $x \leq x \circ a \Rightarrow x \circ -a \leq (x \circ a) \circ -a \Rightarrow x \circ -a \leq x \circ (a \circ -a) \Rightarrow x \circ -a \leq x \circ 0 \Rightarrow x \circ -a \leq 0$;

$[\Leftarrow]$ $x \circ -a \leq 0 \Rightarrow (x \circ a) + (x \circ -a) \leq (x \circ a) + 0 \Rightarrow$

$$x \circ (a + -a) \leq x \circ a \Rightarrow x \circ 1' \leq x \circ a \Rightarrow x \leq x \circ a. \quad \blacksquare$$

With these results and definitions, we will demonstrate the validity of some properties of these algebraic structures.

Definition 3.21 *A partially ordered set (poset) is a pair $\langle P, \leq\rangle$ such that P is a non-empty set and $\leq$ an ordering relation in P, i. e., the relation $\leq$ satisfies the reflexivity, anti-symmetry and transitivity in P.*

Lemma 3.22 *If $\langle P, \leq\rangle$ is a poset, then in $\langle P, \leq\rangle$ the following laws hold:*

(i) *for all $x \in P$, $x \leq a \Rightarrow x \leq b$ if, and only if, $a \leq b$;*

(ii) *for all $x \in P$, $x \leq a \Leftrightarrow x \leq b$ if, and only if, $a = b$;*

(iii) *for all $x \in P$, $b \leq x \Rightarrow a \leq x$ if, and only if, $a \leq b$;*

(iv) *for all $x \in P$, $b \leq x \Leftrightarrow a \leq x$ if, and only if, $a = b$.*

Proof: (i) $[\Rightarrow]$ *Since the condition holds for any x, then it holds for $x = a$.*

Thereby, if $x = a$, then $a \leq a \Rightarrow a \leq b$ and, as $a \leq a$, thus $a \leq b$.

$[\Leftarrow]$ *If $x \leq a$, as $a \leq b$, then $x \leq b$.*

(ii) $[\Rightarrow]$ *Since the condition holds for any x, then it holds for $x = a$ and $x = b$.*

Thereby, if $x = a$, then $a \leq a \Leftrightarrow a \leq b$ and, as $a \leq a$, it is concluded that $a \leq b$.

On the other hand, if $x = b$, then $b \leq a \Leftrightarrow b \leq b$ and, as $b \leq b$, it is concluded that $b \leq a$.

Given that $a \leq b$ and $b \leq a$, we have that $a = b$.

$[\Leftarrow]$ *If $a = b$, then a and b share exactly the same properties and so $x \leq a \Leftrightarrow x \leq b$.*

(iii) $[\Rightarrow]$ *Since the condition holds for all x, then it holds for $x = b$.*

Thereby, if $x = b$, then $b \leq b \Rightarrow a \leq b$ and, as $b \leq b$, thus $a \leq b$.

$[\Leftarrow]$ *If $b \leq x$, as $a \leq b$, then $a \leq x$.*

(iv) $[\Rightarrow]$ *Since the condition holds for all x, then it holds for $x = a$ and $x = b$.*

Thereby, if $x = b$, then $b \leq b \Leftrightarrow a \leq b$ and, as $b \leq b$, it is concluded that $a \leq b$.

On the other hand, if $x = a$, then $b \leq a \Leftrightarrow a \leq a$ and, as $a \leq a$, it is concluded that $b \leq a$.

Given that $a \leq b$ and $b \leq a$, we have that $a = b$.

$[\Leftarrow]$ *If $a = b$, then a and b share exactly the same properties and so $b \leq x \Leftrightarrow a \leq x$.* $\blacksquare$

Proposition 3.23 *In a semiring with modal domain $\langle\langle S, \mathfrak{S}(S)\rangle, \ulcorner\rangle$ the following properties hold:*

(i) $\ulcorner x \leq 0 \Leftrightarrow x \leq 0$;

(ii) *for all $a \in \mathfrak{S}(S)$, $\ulcorner a = a$;*

(iii) $\ulcorner(\ulcorner x) = \ulcorner x$;

(iv) $\ulcorner 0 = 0$;
(v) $\ulcorner(-a) = -\ulcorner a$;
(vi) $x = \ulcorner x \circ x$;
(vii) $x \leq \ulcorner x \circ 1$;
(viii) $\ulcorner(x + y) = \ulcorner x + \ulcorner y$;
(ix) $x \leq y \Rightarrow \ulcorner x \leq \ulcorner y$;
(x) $\ulcorner(x \cdot y) \leq \ulcorner x \cdot \ulcorner y$;
(xi) $\ulcorner(a \circ x) = a \circ \ulcorner x$;
(xii) $\ulcorner(x \circ y) \leq \ulcorner(x \circ \ulcorner y)$;
(xiii) $\ulcorner(x \circ y) = \ulcorner(x \circ \ulcorner y)$;
(xiv) $\ulcorner 1 = 1'$;
(xv) $-\ulcorner x \leq \ulcorner(\sim x)$.

Proof: (i) *As* $x \leq a \circ x \Leftrightarrow \ulcorner x \leq a$*, for* $a = 0$*, we have that* $\ulcorner x \leq 0 \Leftrightarrow x \leq 0 \circ x \Leftrightarrow x \leq 0$.

(ii) *From the Definition 3.12* (i)*, for* $x = a$*, we have that* $a \leq \ulcorner a \circ a$*. Since* a *is a subidentity, then* $a \leq 1'$ *and so* $\ulcorner a \circ a \leq \ulcorner a \circ 1' \Rightarrow \ulcorner a \circ a \leq \ulcorner a$*. As* $a \leq \ulcorner a \circ a$ *and* $\ulcorner a \circ a \leq \ulcorner a$*, it is concluded that* $a \leq \ulcorner a$.

On the other hand, by Definition 3.12 (ii)*,* $\ulcorner(a \circ 1') \leq a$ *and since* $a \circ 1' = a$*, thus* $\ulcorner a \leq a$*. Finally, if* $a \leq \ulcorner a$ *and* $\ulcorner a \leq a$*, then* $\ulcorner a = a$.

(iii) *Given that* $\ulcorner x \in \mathfrak{S}(S)$*, it follows from the item* (ii) *that* $\ulcorner(\ulcorner x) = \ulcorner x$.

(iv) *Since* $0 \in \mathfrak{S}(S)$*, then* $\ulcorner 0 = 0$.

(v) *As* $a, -a \in \mathfrak{S}(S)$*, then, by the item* (ii)*,* $\ulcorner(-a) = -a = -\ulcorner a$.

(vi) *From the Definition 3.12* (i)*, it follows that* $x \leq \ulcorner x \circ x$*. On the other hand, as* $\ulcorner x \in \mathfrak{S}(S)$*, then* $\ulcorner x \leq 1'$ *and thus* $\ulcorner x \circ x \leq 1' \circ x \Rightarrow \ulcorner x \circ x \leq x$*. Thereby,* $\ulcorner x \circ x = x$.

(vii) $x \leq 1 \Rightarrow \ulcorner x \circ x \leq \ulcorner x \circ 1$ *and, as* $x = \ulcorner x \circ x$*, then* $x \leq \ulcorner x \circ 1$.

(viii) *From the Proposition 3.20* (i)*,* $\ulcorner(x + y) \leq a \Leftrightarrow -a \circ (x + y) \leq 0$*. By the lattices, we have that* $-a \circ (x + y) \leq 0 \Leftrightarrow (-a \circ x) + (-a \circ y) \leq 0 \Leftrightarrow -a \circ x \leq 0$ *and* $-a \circ y \leq 0$.

Using again the Proposition 3.20 (i)*, we have that* $-a \circ x \leq 0$ *and* $-a \circ y \leq 0 \Leftrightarrow \ulcorner x \leq a$ *and* $\ulcorner y \leq a \Leftrightarrow \ulcorner x + \ulcorner y \leq a$.

As $\ulcorner(x + y) \leq a \Leftrightarrow \ulcorner x + \ulcorner y \leq a$*, for any* $a \in \mathfrak{S}(S)$*, using the Lemma 3.22* (iv)*, it is concluded that* $\ulcorner(x + y) = \ulcorner x + \ulcorner y$.

(ix) *By the hypothesis we have that* $x \leq y$ *and thus, by the ordering definition,* $x + y = y$*. Thereby,* $x \leq y \Rightarrow x + y = y \Rightarrow \ulcorner(x + y) = \ulcorner y \Rightarrow \ulcorner x + \ulcorner y = \ulcorner y \Rightarrow \ulcorner x \leq \ulcorner y$.

(x) *Since* $x \cdot y \leq x, y$*, the result follows from the item above and from the properties of lattices.*

(xi) *From the condition (ii) of domain definition, it follows that* $\ulcorner(a \circ x) \leq a$*. Since* $a \circ x \leq x$*, then* $\ulcorner(a \circ x) \leq \ulcorner x$ *and so* $\ulcorner(a \circ x) \leq a \cdot \ulcorner x = a \circ \ulcorner x$.

On the other hand, by the definition of modal domain, if we take $x = a$ *and* $y = x$*, then it follows that* $\ulcorner(a \circ \ulcorner x) \leq \ulcorner(a \circ x)$*. As* $a, \ulcorner x \in \mathfrak{S}(S)$*, then* $a \circ \ulcorner x \in \mathfrak{S}(S)$ *and therefore, by the item* (ii)*, it follows that* $\ulcorner(a \circ \ulcorner x) = a \circ \ulcorner x$*. Thereby, it is concluded that* $a \circ \ulcorner x \leq \ulcorner(a \circ x)$.

Lastly, if $\ulcorner(a \circ x) \leq a \circ \ulcorner x$ *and* $a \circ \ulcorner x \leq \ulcorner(a \circ x)$*, then* $\ulcorner(a \circ x) = a \circ \ulcorner x$.

(xii) *Let us see that* $x \circ y = x \circ (\ulcorner y \circ y) = (x \circ \ulcorner y) \circ y = (\ulcorner(x \circ \ulcorner y) \circ (x \circ \ulcorner y)) \circ y = \ulcorner(x \circ \ulcorner y) \circ ((x \circ \ulcorner y) \circ y) = \ulcorner(x \circ \ulcorner y) \circ (x \circ (\ulcorner y \circ y)) = \ulcorner(x \circ \ulcorner y) \circ (x \circ y)$*. Thereby,* $x \circ y \leq \ulcorner(x \circ \ulcorner y) \circ (x \circ y)$.

But, by the Proposition 3.18, $\ulcorner(x \circ y) \leq \ulcorner(x \circ \ulcorner y) \Leftrightarrow x \circ y \leq \ulcorner(x \circ \ulcorner y) \circ (x \circ y)$*. So,* $\ulcorner(x \circ y) \leq \ulcorner(x \circ \ulcorner y)$.

(xiii) *It follows from the previous item and from the condition (iii) of modal domain definition.*

(xiv) *Considering that* $\ulcorner 1 \in \mathfrak{S}(S)$*, then* $\ulcorner 1 \leq 1'$*. On the other hand, as* 1 *is the greatest element of the Boolean semiring, it follows that* $1' \leq 1$ *and, by item* (ix)*,* $\ulcorner 1' \leq \ulcorner 1$*. Given that* $\ulcorner 1' = 1'$*, then* $1' \leq \ulcorner 1$*. Finally, if* $\ulcorner 1 \leq 1'$ *and* $1' \leq \ulcorner 1$*, then* $\ulcorner 1 = 1'$.

(xv) *As* $1' = {}^{\ulcorner}1 = {}^{\ulcorner}(x + \sim x) = {}^{\ulcorner}x + {}^{\ulcorner}(\sim x)$*, then* $1' \leq {}^{\ulcorner}x + {}^{\ulcorner}(\sim x)$*. By the shunting rule* $(x \cdot y \leq z \Leftrightarrow x \leq \sim y + z)$ *which is valid in any complemented distributive lattices with zero and unity, we have:* $1' \leq {}^{\ulcorner}x + {}^{\ulcorner}(\sim x) \Leftrightarrow 1' \cdot (-{}^{\ulcorner}x) \leq {}^{\ulcorner}(\sim x)$ *and so* $1' \cdot (-{}^{\ulcorner}x) \leq {}^{\ulcorner}(\sim x) \Leftrightarrow 1' \circ (-{}^{\ulcorner}x) \leq {}^{\ulcorner}(\sim x) \Leftrightarrow -{}^{\ulcorner}x \leq {}^{\ulcorner}(\sim x)$. ■

Symmetrically, we have the same results for the codomain operator.

Proposition 3.24 *In a semiring with modal codomain* $\langle\langle S, \mathfrak{S}(S)\rangle, {}^{\urcorner}\rangle$ *the following properties hold:*

(i) $x^{\urcorner} \leq 0 \Leftrightarrow x \leq 0$;

(ii) *for all* $a \in \mathfrak{S}(S)$, $a^{\urcorner} = a$;

(iii) $(x^{\urcorner})^{\urcorner} = x^{\urcorner}$;

(iv) $0^{\urcorner} = 0$;

(v) $(-a)^{\urcorner} = -a^{\urcorner}$;

(vi) $x = x \circ x^{\urcorner}$;

(vii) $x \leq 1 \circ x^{\urcorner}$;

(viii) $(x + y)^{\urcorner} = x^{\urcorner} + y^{\urcorner}$;

(ix) $x \leq y \Rightarrow x^{\urcorner} \leq y^{\urcorner}$;

(x) $(x \cdot y)^{\urcorner} \leq x^{\urcorner} \cdot y^{\urcorner}$;

(xi) $(x \circ a)^{\urcorner} = x^{\urcorner} \circ a$;

(xii) $(x \circ y)^{\urcorner} \leq (x^{\urcorner} \circ y)^{\urcorner}$;

(xiii) $(x \circ y)^{\urcorner} = (x^{\urcorner} \circ y)^{\urcorner}$;

(xiv) $1^{\urcorner} = 1'$;

(xv) $-x^{\urcorner} \leq (\sim x)^{\urcorner}$.

Proof: (i) *As* $x^{\urcorner} \leq a \Leftrightarrow x \leq x \circ a$*, for* $a = 0$*, we have that* $x^{\urcorner} \leq 0 \Leftrightarrow x \leq x \circ 0 \Leftrightarrow x \leq 0$.

(ii) *From the Definition 3.14* (i)*, for* $a = x$*, we have that* $a \leq a \circ a^{\urcorner}$*. Since* a *is a subidentity, then* $a \leq 1'$ *and so* $a \circ a^{\urcorner} \leq 1' \circ a^{\urcorner} \Rightarrow a \circ a^{\urcorner} \leq a^{\urcorner}$*. As* $a \leq a \circ a^{\urcorner}$ *and* $a \circ a^{\urcorner} \leq a^{\urcorner}$*, it is concluded that* $a \leq a^{\urcorner}$.

On the other hand, by Definition 3.14 (ii), $(1' \circ a)^{\urcorner} \leq a$ *and since* $1' \circ a = a$*, thus* $a^{\urcorner} \leq a$*. Finally, if* $a \leq a^{\urcorner}$ *and* $a^{\urcorner} \leq a$*, then* $a^{\urcorner} = a$.

(iii) *Given that* $x^{\urcorner} \in \mathfrak{S}(S)$*, it follows from the item* (ii) *that* $(x^{\urcorner})^{\urcorner} = x^{\urcorner}$.

(iv) *Since* $0 \in \mathfrak{S}(S)$*, then* $0^{\urcorner} = 0$.

(v) *As* $a, -a \in \mathfrak{S}(S)$*, then, by the item* (ii), $(-a)^{\urcorner} = -a = -a^{\urcorner}$.

(vi) *From the Definition 3.14* (i)*, it follows that* $x \leq x \circ x^{\urcorner}$*. On the other hand, as* $x^{\urcorner} \in \mathfrak{S}(S)$*, then* $x^{\urcorner} \leq 1'$ *and thus* $x \circ x^{\urcorner} \leq x \circ 1' \Rightarrow x \circ x^{\urcorner} \leq x$*. Thereby,* $x \circ x^{\urcorner} = x$.

(vii) $x \leq 1 \Rightarrow x \circ x^{\urcorner} \leq 1 \circ x^{\urcorner}$ *and, as* $x = x \circ x^{\urcorner}$*, then* $x \leq 1 \circ x^{\urcorner}$.

(viii) *From the Proposition 3.20* (ii), $(x + y)^{\urcorner} \leq a \Leftrightarrow (x + y) \circ -a \leq 0$*. By the lattices, we have that* $(x + y) \circ -a \leq 0 \Leftrightarrow (x \circ -a) + (y \circ -a) \leq 0 \Leftrightarrow x \circ -a \leq 0$ *and* $y \circ -a \leq 0$.

Using again the Proposition 3.20 (ii)*, we have that* $x \circ -a \leq 0$ *and* $y \circ -a \leq 0 \Leftrightarrow x^{\urcorner} \leq a$ *and* $y^{\urcorner} \leq a \Leftrightarrow x^{\urcorner} + y^{\urcorner} \leq a$.

As $(x + y)^{\urcorner} \leq a \Leftrightarrow x^{\urcorner} + y^{\urcorner} \leq a$*, for any* $a \in \mathfrak{S}(S)$*, using the Lemma 3.22* (iv)*, it is concluded that* $(x + y)^{\urcorner} = x^{\urcorner} + y^{\urcorner}$.

(ix) *By the hypothesis we have that* $x \leq y$ *and thus, by the ordering definition,* $x + y = y$*. Thereby,* $x \leq y \Rightarrow x + y = y \Rightarrow (x + y)^{\urcorner} = y^{\urcorner} \Rightarrow x^{\urcorner} + y^{\urcorner} = y^{\urcorner} \Rightarrow x^{\urcorner} \leq y^{\urcorner}$.

(x) *Since* $x \cdot y \leq x, y$*, the result follows from the item above and from the properties of lattices.*

(xi) *From the condition (ii) of codomain definition, it follows that* $(x \circ a)^{\urcorner} \leq a$*. Since* $x \circ a \leq x$*, then* $(x \circ a)^{\urcorner} \leq x^{\urcorner}$ *and so* $(x \circ a)^{\urcorner} \leq x^{\urcorner} \cdot a = x^{\urcorner} \circ a$.

On the other hand, by the definition of modal codomain, if we take $y = a$, then it follows that $(x^{\urcorner} \circ a)^{\urcorner} \leq (x \circ a)^{\urcorner}$. As $a, x^{\urcorner} \in \mathfrak{S}(S)$, then $x^{\urcorner} \circ a \in \mathfrak{S}(S)$ and therefore, by the item (ii), it follows that $(x^{\urcorner} \circ a)^{\urcorner} = x^{\urcorner} \circ a$. Thereby, it is concluded that $x^{\urcorner} \circ a \leq (x \circ a)^{\urcorner}$.

Lastly, if $(x \circ a)^{\urcorner} \leq x^{\urcorner} \circ a$ and $x^{\urcorner} \circ a \leq (x \circ a)^{\urcorner}$, then $(x \circ a)^{\urcorner} = x^{\urcorner} \circ a$.

(xii) *Let us see that $x \circ y = (x \circ x^{\urcorner}) \circ y = x \circ (x^{\urcorner} \circ y) = x \circ ((x^{\urcorner} \circ y) \circ (x^{\urcorner} \circ y)^{\urcorner}) = (x \circ (x^{\urcorner} \circ y)) \circ (x^{\urcorner} \circ y)^{\urcorner} = ((x \circ x^{\urcorner}) \circ y) \circ (x^{\urcorner} \circ y)^{\urcorner} = (x \circ y) \circ (x^{\urcorner} \circ y)^{\urcorner}$. Thereby, $x \circ y \leq (x \circ y) \circ (x^{\urcorner} \circ y)^{\urcorner}$.*

But, by the Proposition 3.19, $(x \circ y)^{\urcorner} \leq (x^{\urcorner} \circ y)^{\urcorner} \Leftrightarrow x \circ y \leq (x \circ y) \circ (x^{\urcorner} \circ y)^{\urcorner}$. So, $(x \circ y)^{\urcorner} \leq (x^{\urcorner} \circ y)^{\urcorner}$.

(xiii) *It follows from the previous item and from the condition (iii) of modal codomain definition.*

(xiv) *Considering that $1^{\urcorner} \in \mathfrak{S}(S)$, then $1^{\urcorner} \leq 1'$. On the other hand, as 1 is the greatest element of the Boolean semiring, it follows that $1' \leq 1$ and, by item (ix), $1'^{\urcorner} \leq 1^{\urcorner}$. Given that $1'^{\urcorner} = 1'$, then $1' \leq 1^{\urcorner}$. Finally, if $1^{\urcorner} \leq 1'$ and $1' \leq 1^{\urcorner}$, then $1^{\urcorner} = 1'$.*

(xv) *As $1' = 1^{\urcorner} = (x + \sim x)^{\urcorner} = x^{\urcorner} + (\sim x)^{\urcorner}$, then $1' \leq x^{\urcorner} + (\sim x)^{\urcorner}$. By the shunting rule ($x \cdot y \leq z \Leftrightarrow x \leq \sim y + z$) which is valid in any complemented distributive lattices with zero and unity, we have: $1' \leq x^{\urcorner} + (\sim x)^{\urcorner} \Leftrightarrow 1' \cdot (-x^{\urcorner}) \leq (\sim x)^{\urcorner}$ and so $1' \cdot (-x^{\urcorner}) \leq (\sim x)^{\urcorner} \Leftrightarrow 1' \circ (-x^{\urcorner}) \leq (\sim x)^{\urcorner} \Leftrightarrow -x^{\urcorner} \leq (\sim x)^{\urcorner}$.* ■

In the next step, we will provide the interpretation of $\mathcal{L}_{PNL}$-formulas into the algebraic structure $\langle\langle S, \mathfrak{S}(S)\rangle, {}^{\ulcorner}, {}^{\urcorner}\rangle$.

4 An algebraic model for $\mathcal{L}_{PNL}$

The possible algebraic model for $\mathcal{L}_{PNL}$ is a structure $\langle\langle S, \mathfrak{S}(S)\rangle, {}^{\ulcorner}, {}^{\urcorner}\rangle$, which is a Boolean semiring with modal bidomain - the structure presented in Section 3.

Thus, $\langle S, \mathfrak{S}(S)\rangle$ is a Boolean semiring with subidentities and ${}^{\ulcorner}$ and ${}^{\urcorner}$ are modal operators of domain and codomain, respectively.

Definition 4.1 *A restrict valuation is a function $\hat{v} : Var(\mathcal{L}_{PNL}) \longrightarrow S$, which interprets each variable of $\mathcal{L}_{PNL}$ into an algebraic element of* **S**.

Definition 4.2 *A valuation is a function $v : For(\mathcal{L}_{PNL}) \longrightarrow S$ which extends $\hat{v}$ to the entire set of $\mathcal{L}_{PNL}$ formulas as follows:*

(i) $v(\neg\varphi) = \sim v(\varphi)$
(ii) $v(\varphi \vee \psi) = v(\varphi) + v(\psi)$
(iii) $v(\varphi \wedge \psi) = v(\varphi) \cdot v(\psi)$
(iv) $v(\varphi \longrightarrow \psi) = \sim v(\varphi) + v(\psi)$
(v) $v(\Diamond_l \varphi) = [v(\varphi)]^{\urcorner} \circ 1 = \delta_l(v(\varphi))$
(vi) $v(\Box_l \varphi) = [-([\sim v(\varphi)]^{\urcorner})] \circ 1 = \omega_l(v(\varphi))$
(vii) $v(\Diamond_r \varphi) = 1 \circ {}^{\ulcorner}[v(\varphi)] = \delta_r(v(\varphi))$
(viii) $v(\Box_r \varphi) = 1 \circ [-({}^{\ulcorner}[\sim v(\varphi)])] = \omega_r(v(\varphi))$.

Definition 4.3 *A formula $\varphi \in For(\mathcal{L}_{PNL})$ is valid in* **S** *if all valuation $v : For(\mathcal{L}_{PNL}) \longrightarrow S$ is a model for φ.*

Definition 4.4 *A valuation $v : For(\mathcal{L}_{PNL}) \longrightarrow S$ is a model for a set of formulas Γ of $\mathcal{L}_{PNL}$ if $v(\varphi) = 1$, for all $\varphi \in \Gamma$.*

When a formula φ is valid in **S**, we will indicate that by $\vDash \varphi$. If $\Gamma \vDash \varphi$, it is understood that φ is a semantic consequence from the set of formulas Γ, and all model of Γ is also a model of φ.

4.1 Soundness between the hilbertian system and the algebraic model

It is desirable that the algebraic model introduced for PNL be (strongly) adequate for this logic. For this, it is necessary to investigate some metatheorems such as Soundness and Completeness theorems.

For this work, we aim to obtain a proof of the Soundness Theorem, thus showing that each axiom or theorem of the Propositional Neighbourhood Logic corresponds to a valid formula in the algebraic structure $\langle\langle S, \mathfrak{S}(S)\rangle, {}^{\ulcorner}, {}^{\urcorner}\rangle$, a Boolean semiring with modal bidomain.

Before we approach the Soundness Theorem, we will show more results necessary for the proof.

Lemma 4.5 $\vDash \varphi \to \psi \Leftrightarrow \upsilon(\varphi) \leq \upsilon(\psi)$.
Proof: Let us remember that $\vDash \varphi \to \psi$ *if, and only if,* $\upsilon(\varphi \to \psi) = 1$. *By the Definition 4.2,* $\upsilon(\varphi \to \psi) = 1 \Leftrightarrow \sim \upsilon(\varphi) + \upsilon(\psi) = 1$. *Now, if* $1 \leq \sim \upsilon(\varphi) + \upsilon(\psi)$, *by the shunting rule* $(x \cdot y \leq z \Leftrightarrow x \leq \sim y + z)$ *which is valid in any complemented distributive lattices with zero and unity, it follows that* $1 \cdot \upsilon(\varphi) \leq \upsilon(\psi)$ *and so* $\upsilon(\varphi) \leq \upsilon(\psi)$. ■

Lemma 4.6 *For* $a \in \mathfrak{S}(S)$ *and* $1 \in S$, *we have that:*
(i) $a \circ 1 \leq (a \circ 1)^{\urcorner} \circ 1$;
(ii) $1 \circ a \leq 1 \circ {}^{\ulcorner}(1 \circ a)$.

Proof: (i) From the Proposition 3.11 (v), $x \leq x \circ 1$ *and so* $a \leq a \circ 1$. *By the monotonicity of* ${}^{\urcorner}$, *it follows that* $a^{\urcorner} \leq (a \circ 1)^{\urcorner}$. *From Proposition 3.24 (ii),* $a^{\urcorner} = a$, *and thus* $a \leq (a \circ 1)^{\urcorner}$. *Then, by Proposition 3.5 (iii), we have that* $a \circ 1 \leq (a \circ 1)^{\urcorner} \circ 1$.

(ii) From the Proposition 3.11 (vi), $x \leq 1 \circ x$ *and so* $a \leq 1 \circ a$. *By the monotonicity of* ${}^{\ulcorner}$, *it follows that* ${}^{\ulcorner}a \leq {}^{\ulcorner}(1 \circ a)$. *From Proposition 3.23 (ii),* ${}^{\ulcorner}a = a$, *and thus* $a \leq {}^{\ulcorner}(1 \circ a)$. *Finally, by Proposition 3.5 (iv), we have that* $1 \circ a \leq 1 \circ {}^{\ulcorner}(1 \circ a)$. ■

In order to understand the following lemmas, as well as the consequences of this result, we indicate some texts about Galois pairs as Dunn and Hardgree (2001), Herrlich and Husek (1990), Ore (1944), Orlowska and Rewitzky (2010) and last but not least Smith (2010).

Lemma 4.7 *The pair* $[\delta_r, \omega_l]$ *is a case of adjunction pair.*
Proof: $[\Rightarrow]$ *Let us take* $x \leq \omega_l(y)$. *Thereby,* $x \leq \omega_l(y) \Rightarrow x \leq -((\sim y)^{\urcorner}) \circ 1 \Rightarrow {}^{\ulcorner}x \leq {}^{\ulcorner}(-((\sim y)^{\urcorner}) \circ 1) \Rightarrow {}^{\ulcorner}x \leq -((\sim y)^{\urcorner}) \circ {}^{\ulcorner}1 \Rightarrow {}^{\ulcorner}x \leq -((\sim y)^{\urcorner}) \circ 1' \Rightarrow {}^{\ulcorner}x \leq -((\sim y)^{\urcorner}) \Rightarrow 1 \circ {}^{\ulcorner}x \leq 1 \circ -((\sim y)^{\urcorner}) \Rightarrow 1 \circ {}^{\ulcorner}x \leq \sim (1 \circ (\sim y)^{\urcorner}) \Rightarrow \sim\sim (1 \circ (\sim y)^{\urcorner}) \leq \sim (1 \circ {}^{\ulcorner}x) \Rightarrow 1 \circ (\sim y)^{\urcorner} \leq \sim (1 \circ {}^{\ulcorner}x)$.

As $\sim y \leq 1 \circ (\sim y)^{\urcorner}$, *then* $\sim y \leq \sim (1 \circ {}^{\ulcorner}x)$. *Finally,* $\sim y \leq \sim (1 \circ {}^{\ulcorner}x) \Rightarrow 1 \circ {}^{\ulcorner}x \leq y \Rightarrow \delta_r(x) \leq y$.

$[\Leftarrow]$ *Let us take* $\delta_r(x) \leq y$. *Thereby,* $\delta_r(x) \leq y \Rightarrow 1 \circ {}^{\ulcorner}x \leq y \Rightarrow \sim y \leq \sim (1 \circ {}^{\ulcorner}x) \Rightarrow \sim y \leq 1 \circ (-{}^{\ulcorner}x) \Rightarrow (\sim y)^{\urcorner} \leq (1 \circ (-{}^{\ulcorner}x))^{\urcorner} \Rightarrow (\sim y)^{\urcorner} \leq 1^{\urcorner} \circ (-{}^{\ulcorner}x) \Rightarrow (\sim y)^{\urcorner} \leq 1' \circ (-{}^{\ulcorner}x) \Rightarrow (\sim y)^{\urcorner} \leq -{}^{\ulcorner}x \Rightarrow - - {}^{\ulcorner}x \leq -((\sim y)^{\urcorner}) \Rightarrow {}^{\ulcorner}x \leq -((\sim y)^{\urcorner}) \Rightarrow {}^{\ulcorner}x \circ 1 \leq -((\sim y)^{\urcorner}) \circ 1 \Rightarrow {}^{\ulcorner}x \circ 1 \leq \omega_l(y)$.

As $x \leq {}^{\ulcorner}x \circ 1$, *then* $x \leq \omega_l(y)$. ■

Lemma 4.8 *The pair* $[\delta_l, \omega_r]$ *is a case of adjunction pair.*
Proof: $[\Rightarrow]$ *Let us take* $x \leq \omega_r(y)$. *Thereby,* $x \leq \omega_r(y) \Rightarrow x \leq 1 \circ -({}^{\ulcorner}(\sim y)) \Rightarrow x^{\urcorner} \leq (1 \circ -({}^{\ulcorner}(\sim y)))^{\urcorner} \Rightarrow x^{\urcorner} \leq 1^{\urcorner} \circ -({}^{\ulcorner}(\sim y)) \Rightarrow x^{\urcorner} \leq 1' \circ -({}^{\ulcorner}(\sim y)) \Rightarrow x^{\urcorner} \leq -({}^{\ulcorner}(\sim y)) \Rightarrow x^{\urcorner} \circ 1 \leq -({}^{\ulcorner}(\sim y)) \circ 1 \Rightarrow x^{\urcorner} \circ 1 \leq \sim ({}^{\ulcorner}(\sim y) \circ 1) \Rightarrow \sim\sim ({}^{\ulcorner}(\sim y) \circ 1) \leq \sim (x^{\urcorner} \circ 1) \Rightarrow {}^{\ulcorner}(\sim y) \circ 1 \leq \sim (x^{\urcorner} \circ 1)$.

As $\sim y \leq {}^{\ulcorner}(\sim y) \circ 1$, *then* $\sim y \leq \sim (x^{\urcorner} \circ 1)$. *Finally,* $\sim y \leq \sim (x^{\urcorner} \circ 1) \Rightarrow x^{\urcorner} \circ 1 \leq y \Rightarrow \delta_l(x) \leq y$.

$[\Leftarrow]$ *Let us take* $\delta_l(x) \leq y$. *Thereby,* $\delta_l(x) \leq y \Rightarrow x^{\urcorner} \circ 1 \leq y \Rightarrow \sim y \leq \sim (x^{\urcorner} \circ 1) \Rightarrow \sim y \leq -x^{\urcorner} \circ 1 \Rightarrow {}^{\ulcorner}(\sim y) \leq {}^{\ulcorner}(-x^{\urcorner} \circ 1) \Rightarrow {}^{\ulcorner}(\sim y) \leq -x^{\urcorner} \circ {}^{\ulcorner}1 \Rightarrow {}^{\ulcorner}(\sim y) \leq -x^{\urcorner} \circ 1' \Rightarrow {}^{\ulcorner}(\sim y) \leq -x^{\urcorner} \Rightarrow - - x^{\urcorner} \leq -{}^{\ulcorner}(\sim y) \Rightarrow x^{\urcorner} \leq -{}^{\ulcorner}(\sim y) \Rightarrow 1 \circ x^{\urcorner} \leq 1 \circ -{}^{\ulcorner}(\sim y)$.

As $x \leq 1 \circ x^{\urcorner}$, *then* $x \leq 1 \circ -{}^{\ulcorner}(\sim y)$ *and so* $x \leq \omega_r(y)$. ■

Theorem 4.9 *(Weak Soundness Theorem)* $\vdash \varphi \Rightarrow \vDash \varphi$.
Proof: Let $\boldsymbol{S} = \langle\langle S, \mathfrak{S}(S)\rangle, ^\ulcorner, ^\urcorner\ \rangle$ *be an algebraic structure of Boolean semiring with subidentities (also a Boolean algebra), equipped with modals operators of domain and codomain.*

In this case, we must show that the axioms are all valid and that the inference rules preserve validity.

(Ax_0) *Since Boolean algebras are algebraic models for CPL, then any CPL-axiom or CPL-theorem is valid in* $\boldsymbol{S}$.

(Ax_1) $\Box_*(\varphi \rightarrow \psi) \rightarrow (\Box_*\varphi \rightarrow \Box_*\psi)$:
Let $[\delta_r, \omega_l]$ *and* $[\delta_l, \omega_r]$ *be pairs of adjunction, stated by the Lemmas 4.7 and 4.8. Thereby,* ω_l *and* ω_r *preserves the order, moreover* $\omega_l(x \cdot y) = \omega_l(x) \cdot \omega_l(y)$ *and* $\omega_r(x \cdot y) = \omega_r(x) \cdot \omega_r(y)$.

Therefore, for any $* \in \{l, r\}$, *it is the case that* $\upsilon(\varphi \rightarrow \psi) \cdot \upsilon(\varphi) \leq \upsilon(\psi) \Rightarrow \omega_*(\upsilon(\varphi \rightarrow \psi) \cdot \upsilon(\varphi)) \leq \omega_*(\upsilon(\psi)) \Rightarrow \omega_*(\upsilon(\varphi \rightarrow \psi)) \cdot \omega_*(\upsilon(\varphi)) \leq \omega_*(\upsilon(\psi))$ *and by the shunting rule* $(x \cdot y \leq z \Leftrightarrow x \leq \sim y + z)$ *we have that* $\omega_*(\upsilon(\varphi \rightarrow \psi)) \leq \sim (\omega_*(\upsilon(\varphi))) + \omega_*(\upsilon(\psi))$ *and so* $\upsilon(\Box_*(\varphi \rightarrow \psi)) \leq \upsilon(\Box_*\varphi \rightarrow \Box_*\psi)$. *Thus* (Ax_1) *is valid in* $\boldsymbol{S}$.

(Ax_2) $\Box_*\varphi \rightarrow \Diamond_*\varphi$:
For $* = l$, *we have that:*

$$
\begin{array}{lrcl}
\textit{Definition 4.2 (iv)}, & \upsilon(\Box_l\varphi \rightarrow \Diamond_l\varphi) & = & \sim \upsilon(\Box_l\varphi) + \upsilon(\Diamond_l\varphi) \\
\textit{Definition 4.2 (vi) and (v)}, & & = & \sim (-([\sim \upsilon(\varphi)]^\urcorner) \circ 1) + ([\upsilon(\varphi)]^\urcorner \circ 1) \\
\sim (a \circ 1) = -a \circ 1, & & = & (- - ([\sim \upsilon(\varphi)]^\urcorner) \circ 1) + ([\upsilon(\varphi)]^\urcorner \circ 1) \\
- - a = a, & & = & ([\sim \upsilon(\varphi)]^\urcorner \circ 1) + ([\upsilon(\varphi)]^\urcorner \circ 1) \\
\textit{Distributivity of } \circ, & & = & ([\sim \upsilon(\varphi)]^\urcorner + [\upsilon(\varphi)]^\urcorner) \circ 1 \\
(x + y)^\urcorner = x^\urcorner + y^\urcorner, & & = & [\sim \upsilon(\varphi) + \upsilon(\varphi)]^\urcorner \circ 1 \\
\sim x + x = 1, & & = & 1^\urcorner \circ 1 \\
1^\urcorner = 1', & & = & 1' \circ 1 \\
1' \textit{ is neutral for } \circ, & & = & 1.
\end{array}
$$

When $* = r$, *it is the case:*

$$
\begin{array}{lrcl}
\textit{Definition 4.2 (iv)}, & \upsilon(\Box_r\varphi \rightarrow \Diamond_r\varphi) & = & \sim \upsilon(\Box_r\varphi) + \upsilon(\Diamond_r\varphi) \\
\textit{Definition 4.2 (viii) and (vii)}, & & = & \sim (1 \circ -(^\ulcorner[\sim \upsilon(\varphi)])) + (1 \circ {}^\ulcorner[\upsilon(\varphi)]) \\
\sim (1 \circ a) = 1 \circ -a, & & = & (1 \circ - - (^\ulcorner[\sim \upsilon(\varphi)])) + (1 \circ {}^\ulcorner[\upsilon(\varphi)]) \\
- - a = a, & & = & (1 \circ {}^\ulcorner[\sim \upsilon(\varphi)]) + (1 \circ {}^\ulcorner[\upsilon(\varphi)]) \\
\textit{Distributivity of } \circ, & & = & 1 \circ (^\ulcorner[\sim \upsilon(\varphi)] + {}^\ulcorner[\upsilon(\varphi)]) \\
^\ulcorner(x + y) = {}^\ulcorner x + {}^\ulcorner y, & & = & 1 \circ {}^\ulcorner[\sim \upsilon(\varphi) + \upsilon(\varphi)] \\
\sim x + x = 1, & & = & 1 \circ {}^\ulcorner 1 \\
^\ulcorner 1 = 1', & & = & 1 \circ 1' \\
1' \textit{ is neutral for } \circ, & & = & 1.
\end{array}
$$

Therefore, the (Ax_2) *is valid in* $\boldsymbol{S}$.

(Ax_3) $\varphi \rightarrow \Box_r\Diamond_l\varphi$ *and* $\varphi \rightarrow \Box_l\Diamond_r\varphi$:
As $[\delta_r, \omega_l]$ *and* $[\delta_l, \omega_r]$ *are pairs of adjunctions, it follows that* $\upsilon(\varphi) \leq \omega_r(\delta_l(\upsilon(\varphi)))$ *and* $\upsilon(\varphi) \leq \omega_l(\delta_r(\upsilon(\varphi)))$. *Thus* $\upsilon(\varphi) \leq \upsilon(\Box_r\Diamond_l\varphi)$ *and* $\upsilon(\varphi) \leq \upsilon(\Box_l\Diamond_r\varphi)$. *Therefore, the* (Ax_3) *is valid*

in **S**.

(Ax_4) $\Diamond_r\Diamond_l\varphi \longrightarrow \Box_r\Diamond_l\varphi$ and $\Diamond_l\Diamond_r\varphi \longrightarrow \Box_l\Diamond_r\varphi$:
For this case, it is sufficient to check that $\upsilon(\Diamond_r\Diamond_l\varphi) = \upsilon(\Box_r\Diamond_l\varphi)$ and $\upsilon(\Diamond_l\Diamond_r\varphi) = \upsilon(\Box_l\Diamond_r\varphi)$.

$$\begin{array}{rcl} \upsilon(\Diamond_r\Diamond_l\varphi) & = & 1 \circ {}^{\ulcorner}(\upsilon(\Diamond_l\varphi)) \\ & = & 1 \circ {}^{\ulcorner}(\upsilon(\varphi)^{\urcorner} \circ 1) \\ & = & 1 \circ (\upsilon(\varphi)^{\urcorner} \circ {}^{\ulcorner}1) \\ & = & 1 \circ (\upsilon(\varphi)^{\urcorner} \circ 1') \\ & = & 1 \circ \upsilon(\varphi)^{\urcorner} \end{array} \quad \textit{and} \quad \begin{array}{rcl} \upsilon(\Box_r\Diamond_l\varphi) & = & 1 \circ -{}^{\ulcorner}(\sim \upsilon(\Diamond_l\varphi)) \\ & = & 1 \circ -{}^{\ulcorner}(\sim (\upsilon(\varphi)^{\urcorner} \circ 1)) \\ & = & 1 \circ -{}^{\ulcorner}(-\upsilon(\varphi)^{\urcorner} \circ 1) \\ & = & 1 \circ -(-\upsilon(\varphi)^{\urcorner} \circ {}^{\ulcorner}1) \\ & = & 1 \circ -(-\upsilon(\varphi)^{\urcorner} \circ 1') \\ & = & 1 \circ -(-\upsilon(\varphi)^{\urcorner}) \\ & = & 1 \circ \upsilon(\varphi)^{\urcorner} \end{array}$$

$$\begin{array}{rcl} \upsilon(\Diamond_l\Diamond_r\varphi) & = & (\upsilon(\Diamond_r\varphi))^{\urcorner} \circ 1 \\ & = & (1 \circ {}^{\ulcorner}\upsilon(\varphi))^{\urcorner} \circ 1 \\ & = & (1^{\urcorner} \circ {}^{\ulcorner}\upsilon(\varphi)) \circ 1 \\ & = & (1' \circ {}^{\ulcorner}\upsilon(\varphi)) \circ 1 \\ & = & {}^{\ulcorner}\upsilon(\varphi) \circ 1 \end{array} \quad \textit{and} \quad \begin{array}{rcl} \upsilon(\Box_l\Diamond_r\varphi) & = & -(\sim \upsilon(\Diamond_r\varphi))^{\urcorner} \circ 1 \\ & = & -(\sim (1 \circ {}^{\ulcorner}\upsilon(\varphi)))^{\urcorner} \circ 1 \\ & = & -(1 \circ -{}^{\ulcorner}\upsilon(\varphi))^{\urcorner} \circ 1 \\ & = & -(1^{\urcorner} \circ -{}^{\ulcorner}\upsilon(\varphi)) \circ 1 \\ & = & -(1' \circ -{}^{\ulcorner}\upsilon(\varphi)) \circ 1 \\ & = & -(-{}^{\ulcorner}\upsilon(\varphi)) \circ 1 \\ & = & {}^{\ulcorner}\upsilon(\varphi) \circ 1 \end{array}$$

Thereby, $\upsilon(\Diamond_r\Diamond_l\varphi) \leq \upsilon(\Box_r\Diamond_l\varphi)$ and $\upsilon(\Diamond_l\Diamond_r\varphi) \leq \upsilon(\Box_l\Diamond_r\varphi)$. It follows that $(Ax4)$ is also valid in **S**.

(Ax_5) $\Diamond_*\Diamond_*\Diamond_*\varphi \longrightarrow \Diamond_*\Diamond_*\varphi$:
Let us see that $\upsilon(\Diamond_l\Diamond_l\Diamond_l\varphi) = \upsilon(\Diamond_l\Diamond_l\varphi)$ and $\upsilon(\Diamond_r\Diamond_r\Diamond_r\varphi) = \upsilon(\Diamond_r\Diamond_r\varphi)$.

$$\begin{array}{rl} & \upsilon(\Diamond_l\Diamond_l\Diamond_l\varphi) \\ = & ((\upsilon(\varphi)^{\urcorner} \circ 1)^{\urcorner} \circ 1)^{\urcorner} \circ 1 \\ = & ((\upsilon(\varphi) \circ 1)^{\urcorner} \circ 1)^{\urcorner} \circ 1 \\ = & ((\upsilon(\varphi) \circ 1) \circ 1)^{\urcorner} \circ 1 \\ = & (\upsilon(\varphi) \circ (1 \circ 1))^{\urcorner} \circ 1 \\ = & (\upsilon(\varphi) \circ 1)^{\urcorner} \circ 1 \\ = & (\upsilon(\varphi)^{\urcorner} \circ 1)^{\urcorner} \circ 1 \\ = & \upsilon(\Diamond_l\Diamond_l\varphi) \end{array} \quad \textit{and} \quad \begin{array}{rl} & \upsilon(\Diamond_r\Diamond_r\Diamond_r\varphi) \\ = & 1 \circ {}^{\ulcorner}(1 \circ {}^{\ulcorner}(1 \circ {}^{\ulcorner}\upsilon(\varphi))) \\ = & 1 \circ {}^{\ulcorner}(1 \circ {}^{\ulcorner}(1 \circ \upsilon(\varphi))) \\ = & 1 \circ {}^{\ulcorner}(1 \circ (1 \circ \upsilon(\varphi))) \\ = & 1 \circ {}^{\ulcorner}((1 \circ 1) \circ \upsilon(\varphi)) \\ = & 1 \circ {}^{\ulcorner}(1 \circ \upsilon(\varphi)) \\ = & 1 \circ {}^{\ulcorner}(1 \circ {}^{\ulcorner}\upsilon(\varphi)) \\ = & \upsilon(\Diamond_r\Diamond_r\varphi) \end{array}$$

So, $\upsilon(\Diamond_l\Diamond_l\Diamond_l\varphi) \leq \upsilon(\Diamond_l\Diamond_l\varphi)$ and $\upsilon(\Diamond_r\Diamond_r\Diamond_r\varphi) \leq \upsilon(\Diamond_r\Diamond_r\varphi)$. Thus, the (Ax_5) is also valid in **S**.

(Ax_6) $\Box_r\Diamond_l\varphi \longrightarrow \Diamond_l\Diamond_r\Diamond_r\varphi \vee \Diamond_l\Diamond_l\Diamond_r\varphi$ and $\Box_l\Diamond_r\varphi \longrightarrow \Diamond_r\Diamond_l\Diamond_l\varphi \vee \Diamond_r\Diamond_r\Diamond_l\varphi$:
$x \leq 1 \circ x \Rightarrow {}^{\ulcorner}\upsilon(\varphi) \leq 1 \circ {}^{\ulcorner}\upsilon(\varphi) \Rightarrow {}^{\ulcorner\ulcorner}\upsilon(\varphi) \leq {}^{\ulcorner}(1 \circ {}^{\ulcorner}\upsilon(\varphi)) \Rightarrow {}^{\ulcorner}\upsilon(\varphi) \leq {}^{\ulcorner}(1 \circ {}^{\ulcorner}\upsilon(\varphi)) \Rightarrow 1 \circ {}^{\ulcorner}\upsilon(\varphi) \leq 1 \circ {}^{\ulcorner}(1 \circ {}^{\ulcorner}\upsilon(\varphi))$. Given that $1 \circ {}^{\ulcorner}\upsilon(\varphi) = \upsilon(\Diamond_r\varphi)$ so it follows that $\upsilon(\Diamond_r\varphi) \leq \upsilon(\Diamond_r\Diamond_r\varphi)$.

On the other hand, $x \leq x \circ 1 \Rightarrow \upsilon(\varphi)^{\urcorner} \leq \upsilon(\varphi)^{\urcorner} \circ 1 \Rightarrow \upsilon(\varphi)^{\urcorner\urcorner} \leq (\upsilon(\varphi)^{\urcorner} \circ 1)^{\urcorner} \Rightarrow \upsilon(\varphi)^{\urcorner} \leq (\upsilon(\varphi)^{\urcorner} \circ 1)^{\urcorner} \Rightarrow \upsilon(\varphi)^{\urcorner} \circ 1 \leq (\upsilon(\varphi)^{\urcorner} \circ 1)^{\urcorner} \circ 1$. Given that $\upsilon(\varphi)^{\urcorner} \circ 1 = \upsilon(\Diamond_l\varphi)$, it follows that $\upsilon(\Diamond_l\varphi) \leq \upsilon(\Diamond_l\Diamond_l\varphi)$.

By the monotonicity of δ_l, a consequence of the pairs of adjunctions, then follows that $\upsilon(\Diamond_l\Diamond_r\varphi) \leq \upsilon(\Diamond_l\Diamond_r\Diamond_r\varphi)$. As $\upsilon(\Diamond_l\varphi) \leq \upsilon(\Diamond_l\Diamond_l\varphi)$, then $\upsilon(\Diamond_l\Diamond_r\varphi) \leq \upsilon(\Diamond_l\Diamond_l\Diamond_r\varphi)$. And so $\upsilon(\Diamond_l\Diamond_r\varphi) + \upsilon(\Diamond_l\Diamond_r\varphi) \leq \upsilon(\Diamond_l\Diamond_r\Diamond_r\varphi) + \upsilon(\Diamond_l\Diamond_l\Diamond_r\varphi)$. By the idempotence of $+$, then $\upsilon(\Diamond_l\Diamond_r\varphi) \leq \upsilon(\Diamond_l\Diamond_r\Diamond_r\varphi) + \upsilon(\Diamond_l\Diamond_l\Diamond_r\varphi)$.

Lastly, it remains to show that $\upsilon(\Box_r\Diamond_l\varphi) \leq \upsilon(\Diamond_l\Diamond_r\varphi)$.

Let us see that $x \leq 1 \circ x^{\urcorner} \Rightarrow {}^{\ulcorner}x \circ x \leq {}^{\ulcorner}x \circ (1 \circ x^{\urcorner}) \Rightarrow x \leq ({}^{\ulcorner}x \circ 1) \circ x^{\urcorner}$. *But* $({}^{\ulcorner}x \circ 1) \circ x^{\urcorner} \leq {}^{\ulcorner}x \circ 1$, *since* $x^{\urcorner} \leq 1 \Rightarrow 1 \circ x^{\urcorner} \leq 1 \circ 1 \Rightarrow {}^{\ulcorner}x \circ (1 \circ x^{\urcorner}) \leq {}^{\ulcorner}x \circ 1 \Rightarrow ({}^{\ulcorner}x \circ 1) \circ x^{\urcorner} \leq {}^{\ulcorner}x \circ 1$. *Thereby,* $x \leq {}^{\ulcorner}x \circ 1$. *As* $x \leq 1 \circ x^{\urcorner} \Rightarrow x \leq {}^{\ulcorner}x \circ 1$, *it follows from the Lemma 3.22 that* $1 \circ x^{\urcorner} \leq {}^{\ulcorner}x \circ 1$. *In particular,* $1 \circ \upsilon(\varphi)^{\urcorner} \leq {}^{\ulcorner}\upsilon(\varphi) \circ 1$. *Already seen in the proof of the validity of* $(Ax4)$, *we have that* $1 \circ \upsilon(\varphi)^{\urcorner} = \upsilon(\Box_r \Diamond_l \varphi)$ *and* ${}^{\ulcorner}\upsilon(\varphi) \circ 1 = \upsilon(\Diamond_l \Diamond_r \varphi)$, *then it follows that* $\upsilon(\Box_r \Diamond_l \varphi) \leq \upsilon(\Diamond_l \Diamond_r \varphi)$.

Therefore, as $\upsilon(\Box_r \Diamond_l \varphi) \leq \upsilon(\Diamond_l \Diamond_r \varphi)$ *and* $\upsilon(\Diamond_l \Diamond_r \varphi) \leq \upsilon(\Diamond_l \Diamond_r \Diamond_r \varphi) + \upsilon(\Diamond_l \Diamond_l \Diamond_r \varphi)$, *then* $\upsilon(\Box_r \Diamond_l \varphi) \leq \upsilon(\Diamond_l \Diamond_r \Diamond_r \varphi) + \upsilon(\Diamond_l \Diamond_l \Diamond_r \varphi)$ *and the formula is valid in* $\boldsymbol{S}$.

Similarly, we show the validity of the second formula of $(Ax6)$. *As already seen,* $\upsilon(\Diamond_r \varphi) \leq \upsilon(\Diamond_r \Diamond_r \varphi)$ *and* $\upsilon(\Diamond_l \varphi) \leq \upsilon(\Diamond_l \Diamond_l \varphi)$. *Thus, it follows that* $\upsilon(\Diamond_r \Diamond_l \varphi) \leq \upsilon(\Diamond_r \Diamond_r \Diamond_l \varphi)$ *and, by the monotonicity of* δ_r, $\upsilon(\Diamond_r \Diamond_l \varphi) \leq \upsilon(\Diamond_r \Diamond_l \Diamond_l \varphi)$. *Therefore,* $\upsilon(\Diamond_r \Diamond_l \varphi) \leq \upsilon(\Diamond_r \Diamond_l \Diamond_l \varphi) + \upsilon(\Diamond_r \Diamond_r \Diamond_l \varphi)$.

It remains to show that $\upsilon(\Box_l \Diamond_r \varphi) \leq \upsilon(\Diamond_r \Diamond_l \varphi)$. *Let us see that* $x \leq {}^{\ulcorner}x \circ 1 \Rightarrow x \circ x^{\urcorner} \leq ({}^{\ulcorner}x \circ 1) \circ x^{\urcorner} \Rightarrow x \leq {}^{\ulcorner}x \circ (1 \circ x^{\urcorner})$. *But* ${}^{\ulcorner}x \circ (1 \circ x^{\urcorner}) \leq 1 \circ x^{\urcorner}$, *since* ${}^{\ulcorner}x \leq 1 \Rightarrow {}^{\ulcorner}x \circ 1 \leq 1 \circ 1 \Rightarrow ({}^{\ulcorner}x \circ 1) \circ x^{\urcorner} \leq 1 \circ x^{\urcorner} \Rightarrow {}^{\ulcorner}x \circ (1 \circ x^{\urcorner}) \leq 1 \circ x^{\urcorner}$. *Thereby,* $x \leq 1 \circ x^{\urcorner}$. *As* $x \leq {}^{\ulcorner}x \circ 1 \Rightarrow x \leq 1 \circ x^{\urcorner}$, *it follows from the Lemma 3.22 that* ${}^{\ulcorner}x \circ 1 \leq 1 \circ x^{\urcorner}$. *In particular,* ${}^{\ulcorner}\upsilon(\varphi) \circ 1 \leq 1 \circ \upsilon(\varphi)^{\urcorner}$. *Let* ${}^{\ulcorner}\upsilon(\varphi) \circ 1 = \upsilon(\Box_l \Diamond_r \varphi)$ *and* $1 \circ \upsilon(\varphi)^{\urcorner} = \upsilon(\Diamond_r \Diamond_l \varphi)$, *we obtain that* $\upsilon(\Box_l \Diamond_r \varphi) \leq \upsilon(\Diamond_r \Diamond_l \varphi)$.

Therefore, as $\upsilon(\Box_l \Diamond_r \varphi) \leq \upsilon(\Diamond_r \Diamond_l \varphi)$ *and* $\upsilon(\Diamond_r \Diamond_l \varphi) \leq \upsilon(\Diamond_r \Diamond_l \Diamond_l \varphi) + \upsilon(\Diamond_r \Diamond_r \Diamond_l \varphi)$, *then* $\upsilon(\Box_l \Diamond_r \varphi) \leq \upsilon(\Diamond_r \Diamond_l \Diamond_l \varphi) + \upsilon(\Diamond_r \Diamond_r \Diamond_l \varphi)$.

Thus, the $(Ax6)$ *is valid in* $\boldsymbol{S}$.

(Ax_7) $(\Box_l \varphi \wedge \Diamond_l \psi_1 \wedge \ldots \wedge \Diamond_l \psi_n) \longrightarrow \Diamond_l(\Box_l \varphi \wedge \Diamond_l \psi_1 \wedge \ldots \wedge \Diamond_l \psi_n)$ *and*
$(\Box_r \varphi \wedge \Diamond_r \psi_1 \wedge \ldots \wedge \Diamond_r \psi_n) \longrightarrow \Diamond_r(\Box_r \varphi \wedge \Diamond_r \psi_1 \wedge \ldots \wedge \Diamond_r \psi_n)$*:*

From an algebraic point of view, we must show that $\upsilon(\Box_l \varphi \wedge \Diamond_l \psi_1 \wedge \ldots \wedge \Diamond_l \psi_n) \leq \upsilon(\Diamond_l(\Box_l \varphi \wedge \Diamond_l \psi_1 \wedge \ldots \wedge \Diamond_l \psi_n)) = (\upsilon(\Box_l \varphi \wedge \Diamond_l \psi_1 \wedge \ldots \wedge \Diamond_l \psi_n))^{\urcorner} \circ 1$ *and* $\upsilon(\Box_r \varphi \wedge \Diamond_r \psi_1 \wedge \ldots \wedge \Diamond_r \psi_n) \leq \upsilon(\Diamond_r(\Box_r \varphi \wedge \Diamond_r \psi_1 \wedge \ldots \wedge \Diamond_r \psi_n)) = 1 \circ {}^{\ulcorner}(\upsilon(\Box_r \varphi \wedge \Diamond_r \psi_1 \wedge \ldots \wedge \Diamond_r \psi_n))$.

But $\upsilon(\Box_l \varphi \wedge \Diamond_l \psi_1 \wedge \ldots \wedge \Diamond_l \psi_n) = \upsilon(\Box_l \varphi) \cdot \upsilon(\Diamond_l \psi_1) \cdot \ldots \cdot \upsilon(\Diamond_l \psi_n) = (-(\sim \upsilon(\varphi))^{\urcorner} \circ 1) \cdot (\upsilon(\psi_1)^{\urcorner} \circ 1) \cdot \ldots \cdot (\upsilon(\psi_n)^{\urcorner} \circ 1) = (-(\sim \upsilon(\varphi))^{\urcorner} \cdot \upsilon(\psi_1)^{\urcorner} \cdot \ldots \cdot \upsilon(\psi_n)^{\urcorner}) \circ 1$ *and* $\upsilon(\Box_r \varphi \wedge \Diamond_r \psi_1 \wedge \ldots \wedge \Diamond_r \psi_n) = \upsilon(\Box_r \varphi) \cdot \upsilon(\Diamond_r \psi_1) \cdot \ldots \cdot \upsilon(\Diamond_r \psi_n) = (1 \circ -{}^{\ulcorner}(\sim \upsilon(\varphi))) \cdot (1 \circ {}^{\ulcorner}\upsilon(\psi_1)) \cdot \ldots \cdot (1 \circ {}^{\ulcorner}\upsilon(\psi_n)) = 1 \circ (-{}^{\ulcorner}(\sim \upsilon(\varphi)) \cdot {}^{\ulcorner}\upsilon(\psi_1) \cdot \ldots \cdot {}^{\ulcorner}\upsilon(\psi_n))$.

Since $(-(\sim \upsilon(\varphi))^{\urcorner} \cdot \upsilon(\psi_1)^{\urcorner} \cdot \ldots \cdot \upsilon(\psi_n)^{\urcorner})$ *and* $(-{}^{\ulcorner}(\sim \upsilon(\varphi)) \cdot {}^{\ulcorner}\upsilon(\psi_1) \cdot \ldots \cdot {}^{\ulcorner}\upsilon(\psi_n))$ *are subidentities, then by the Lemma 4.6 we have that the* $(Ax7)$ *is valid in* $\boldsymbol{S}$ *for any* $1 \leq n$*: for Lemma 4.6 item (i) just take* $a = (-(\sim \upsilon(\varphi))^{\urcorner} \cdot \upsilon(\psi_1)^{\urcorner} \cdot \ldots \cdot \upsilon(\psi_n)^{\urcorner})$ *and for Lemma 4.6 item (ii) just take* $a = (-{}^{\ulcorner}(\sim \upsilon(\varphi)) \cdot {}^{\ulcorner}\upsilon(\psi_1) \cdot \ldots \cdot {}^{\ulcorner}\upsilon(\psi_n))$.

(MP)*: If* $\vDash \varphi$ *and* $\vDash \varphi \rightarrow \psi$, *then* $\upsilon(\varphi) = 1$ *and* $\upsilon(\varphi \rightarrow \psi) = 1$. *It follows that* $\upsilon(\varphi) \leq \upsilon(\psi) \Rightarrow 1 \leq \upsilon(\psi)$ *and so* $\upsilon(\psi) = 1$. *Thereby, the rule* (MP) *preserve validity.*

(RN)*: As* $\vDash \varphi$, *then* $\upsilon(\varphi) = 1$. *Thus,* $[-([\sim 1]^{\urcorner})] \circ 1 = [-(0^{\urcorner})] \circ 1 = [-0] \circ 1 = 1' \circ 1 = 1$. *It is also the case that* $1 \circ [-({}^{\ulcorner}[\sim 1])] = 1 \circ [-({}^{\ulcorner}[0])] = 1 \circ [-0] = 1 \circ 1' = 1$. *So the rule* (RN) *preserve validity.* ■

Corollary 4.10 *(Strong Soundness Theorem)* $\Gamma \vdash \varphi \Rightarrow \Gamma \vDash \varphi$.
Proof: The proof follows by induction on the length of the deduction $\Gamma \vdash \varphi$.

If $\Gamma \vdash \varphi$ *has an only step, then* φ *is an axiom or* $\varphi \in \Gamma$. *When* φ *is an axiom, then by the previous theorem it is valid in* $\boldsymbol{S}$. *If* $\varphi \in \Gamma$, *all model of* Γ *is also a model of* φ.

But, if the deduction has more than one step, then the formula is obtained by the rules of inference, rules which preserve validity by the previous theorem. ■

5 Conclusions

This work is part of an academic research, which the main objective is to obtain an algebraic model adequate for a case of temporal modal logic. To obtain the adequacy, two paths are necessary: the Soundness, to infer that every axiom, theorem and deduction are valid in the respective algebraic model, and Completeness, to obtain whether every validity obtained in the algebra corresponds to some correct deduction in the axiomatic system of PNL.

The aim of this work was to develop part of the path towards the adequacy, in this case, the Soundness. Finally, with the results obtained, it can be said that the algebraic model for PNL is, at least, Sound, in relation to its deductive system.

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