Revista Eletrônica
Paulista de Matemática

ISSN 2316-9664
v. 25, 2025
Edição Ermac

A characterization of adjunction in a many-valued modal system

Uma caracterização de adjunção em um sistema modal multivalorado

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Abstract

Galois connections are pairs of functions, defined over ordered sets, that preserve some particular aspects. They are studied in the context of algebraic structures. For the present work, we consider a four-valued logic called PM4N, which includes at least two modal operators. In this paper, we develop a particular four-valued implication for PM4N, which constitutes a Galois pair with the two modal operators. Then, we show some properties of the logic just considering the correlates algebraic developments.

Keywords: Galois connection. Many-valued logic. Modal logic. Algebraic logic.

Resumo

Conexões de Galois são pares de funções, definidas sobre conjuntos ordenados, que preservam alguns aspectos particulares. Elas são estudadas no contexto das estruturas algébricas. Consideramos, para o presente trabalho, uma lógica de quatro valores chamada PM4N, que contempla, pelo menos, dois operadores modais. Neste artigo, desenvolvemos uma particular implicação para o sistema de quatro valores PM4N, que se constitui em um par de Galois para estes dois operadores modais. Então, mostramos algumas propriedades da lógica que decorrem dos desenvolvimentos algébricos dos pares de Galois.

Palavras-chave: Conexões de Galois. Lógica polivalente. Lógica modal. Lógica algébrica.

Artigo recebido em ago. 2024 e aceito em abr. 2025

1 Introduction

This article develops an interaction between logic and algebra. It extends the paper [1] by the inclusion of some theoretical results and the proofs of the central results.

We take the logic PM4N, introduced by [2] as a four-valued logic, that contains also some modal operators, central for the modal logics [3], [4] and [5]. These two operators are two unary functions defined in the set of formulas of PM4N. To understand some aspects of this logic, we need to grasp properties of these operators.

By verifying some properties of PM4N, we have considered that the pair of modal operators could be a Galois pair. It was not straightforward to characterize the Galois pair. For that, we needed to obtain another connective of implication or, in algebraic notion, another ordering relation.

So we begin with a short presentation of PM4N and in the following some theoretic elements about the Galois pairs. As an original contribution, we look for a particular implication that makes the modal operators of necessary and possible a Galois adjunction. Then, using the elements of Galois connections, we can show some properties of these modal operators.

2 The logic PM4N

The logic PM4N is generated from the operators $L = \{\vee, \neg, \Box\}$, such that $\vee$, $\neg$ and $\Box$ denote, respectively, the notions of disjunction, negation and necessity. We use the same symbols for the propositional language and for the four-valued matrix semantic of PM4N.

The four values and the matrix semantics is the following:

$$\mathcal{M}_{PM4N} = (\{0, n, b, 1\}, \vee, \neg, \Box, \{b, 1\}),$$

such that b and 1 are the designated values and 0 and n are the non-designated values. The set of designated or true values is denoted by $D = \{b, 1\}$.

We must think about these elements as a Boolean algebra of four elements in this disposition:

$$\begin{array}{ccc} & 1 & \\ \diagup & & \diagdown \\ n & & b \\ \diagdown & & \diagup \\ & 0 & \end{array}$$

The disjunction corresponds to the supremum, that is, $x \vee y = sup\{x, y\}$. The negation corresponds to the boolean complement and the necessitation that only the value 1 is necessarily true. So, we have the following tables:

$\vee$	0	n	b	1
0	0	n	b	1
n	n	n	1	1
b	b	1	b	1
1	1	1	1	1

	$\neg$
0	1
n	b
b	n
1	0

	$\Box$
0	0
n	0
b	0
1	1

FEITOSA, H. A.; FREITAS, R. A. A characterization of adjunction in a many-valued modal system. **C.Q.D. – Revista Eletrônica Paulista de Matemática**, Bauru, v. 25, e25005, 2025. Edição Ermac.
DOI: 10.21167/cqdv25e25005 Disponível em: https://sistemas.fc.unesp.br/ojs/index.php/revistacqd/index

We have a four-valued logic and, as usual, the values 0 and 1 represent, respectively, the *falsun* and the *verun*, but b and n are two complementary values that are not in a linear order.

The $For(PM4N)$ must be understood as the set of formulas of the PM4N language. We also evoke PM4N-valuation v for the valuation function $v : For(PM4N) \longrightarrow \{0, n, b, 1\}$ which maps formulas into one of the four values of the matrix semantics of PM4N.

Definition 2.1 *A formula $\varphi \in For(PM4N)$ is valid, according to $\mathcal{M}_{PM4N}$, if, for every PM4N-valuation v, $v(\varphi) \in D$.*

Definition 2.2 *If $\Gamma \cup \{\varphi\} \subseteq For(PM4N)$, then Γ implies logically φ, or φ is a semantic consequence of Γ, when, for every PM4N-valuation v, if $v(\Gamma) \subseteq D$, then $v(\varphi) \in D$.*

Thus, we have that for every valuation v:

$$\Gamma \vDash \varphi \iff (v(\Gamma) \subseteq D \Rightarrow v(\varphi) \in D).$$

It is possible to define the following operators in PM4N:

Possibility: $\Diamond x =_{def} \neg\Box\neg x$
Conjunction: $x \wedge y =_{def} \neg(\neg x \vee \neg y)$
Conditional: $x \longrightarrow y =_{def} \neg x \vee y$
Biconditional: $x \leftrightarrow y =_{def} (x \longrightarrow y) \wedge (y \longrightarrow x)$
Consistency: $\circ x =_{def} \Box x \vee \neg\Diamond x$
Inconsistency (or Contingency): $\bullet x =_{def} \Diamond x \wedge \neg\Box x$
Paraconsistent negation: $\sim x =_{def} \neg x \leftrightarrow \circ x$

The meanings of these operators are given by the following tables:

	$\Diamond$
0	0
n	1
b	1
1	1

$\wedge$	0	n	b	1
0	0	0	0	0
n	0	n	0	n
b	0	0	b	b
1	0	n	b	1

$\longrightarrow$	0	n	b	1
0	1	1	1	1
n	b	1	b	1
b	n	n	1	1
1	0	n	b	1

$\leftrightarrow$	0	n	b	1
0	1	b	n	0
n	b	1	0	n
b	n	0	1	b
1	0	n	b	1

	$\circ$
0	1
n	0
b	0
1	1

	$\bullet$
0	0
n	1
b	1
1	0

	$\sim$
0	1
n	n
b	b
1	0

3 A deductive system for PM4N

From the matrix model of PM4N, we have elaborated a completely adequate tableaux system for this logic [6].

We will consider the following: (i) $f \in \{0, n\}$ for false values and $t \in \{b, 1\}$ for true values, (ii) the use of signed formulas as $k \ \ \psi$ to indicate that the formula ψ has a value $k \in \{0, n, b, 1\}$ and (iii) we will denote the Boolean complement of $k \in \{0, n, b, 1\}$ by k'.

We present below the tableau rules and some definitions of [6].

$f-t$ expansions:

$$\frac{f \quad \varphi}{0\ \varphi \;\mid\; n\ \varphi} \qquad\qquad \frac{t \quad \varphi}{b\ \varphi \;\mid\; 1\ \varphi}$$

Negation:

$$\frac{k \quad \neg\varphi}{k' \quad \varphi}$$

Conjunction:

$$\frac{0 \quad \varphi\wedge\psi}{0\ \varphi \;\mid\; 0\ \psi \;\mid\; \begin{array}{c} b\ \varphi \\ n\ \psi \end{array} \;\mid\; \begin{array}{c} n\ \varphi \\ b\ \psi \end{array}} \qquad\qquad \frac{n \quad \varphi\wedge\psi}{\begin{array}{ccccc} \multicolumn{3}{c}{n\ \varphi} & \mid & 1\ \varphi \\ n\ \psi & \mid & 1\ \psi & \mid & n\ \psi \end{array}}$$

$$\frac{b \quad \varphi\wedge\psi}{\begin{array}{c} b\ \varphi \\ t\ \psi \end{array} \;\mid\; \begin{array}{c} 1\ \varphi \\ b\ \psi \end{array}} \qquad\qquad \frac{1 \quad \varphi\wedge\psi}{\begin{array}{c} 1\ \varphi \\ 1\ \psi \end{array}}$$

Disjunction:

$$\frac{0 \quad \varphi\vee\psi}{\begin{array}{c} 0\ \varphi \\ 0\ \psi \end{array}} \qquad\qquad \frac{n \quad \varphi\vee\psi}{\begin{array}{c} n\ \varphi \\ f\ \psi \end{array} \;\mid\; \begin{array}{c} 0\ \varphi \\ n\ \psi \end{array}} \qquad\qquad \frac{b \quad \varphi\vee\psi}{\begin{array}{ccccc} \multicolumn{3}{c}{b\ \varphi} & \mid & 0\ \varphi \\ b\ \psi & \mid & 0\ \psi & \mid & b\ \psi \end{array}}$$

$$\frac{1 \quad \varphi\vee\psi}{1\ \varphi \;\mid\; 1\ \psi \;\mid\; \begin{array}{c} b\ \varphi \\ n\ \psi \end{array} \;\mid\; \begin{array}{c} n\ \varphi \\ b\ \psi \end{array}}$$

Conditional:

$$\frac{0 \quad \varphi\rightarrow\psi}{\begin{array}{c} 1\ \varphi \\ 0\ \psi \end{array}} \qquad\qquad \frac{n \quad \varphi\rightarrow\psi}{\begin{array}{c} b\ \varphi \\ f\ \psi \end{array} \;\mid\; \begin{array}{c} 1\ \varphi \\ n\ \psi \end{array}} \qquad\qquad \frac{b \quad \varphi\rightarrow\psi}{\begin{array}{ccccc} \multicolumn{3}{c}{n\ \varphi} & \mid & 1\ \varphi \\ 0\ \psi & \mid & b\ \psi & \mid & b\ \psi \end{array}}$$

$$\frac{1 \quad \varphi\rightarrow\psi}{0\ \varphi \;\mid\; 1\ \psi \;\mid\; \begin{array}{c} n\ \varphi \\ n\ \psi \end{array} \;\mid\; \begin{array}{c} b\ \varphi \\ b\ \psi \end{array}}$$

Modal operators:

$$\frac{1 \quad \Box\varphi}{1 \quad \varphi} \qquad \frac{0 \quad \Diamond\varphi}{0 \quad \varphi} \qquad \frac{0 \quad \Box\varphi}{f\ \varphi \;\mid\; b\ \varphi} \qquad \frac{1 \quad \Diamond\varphi}{n\ \varphi \;\mid\; t\ \varphi}$$

Definition 3.1 *A branch of a tableau of $\mathcal{T}_{PM4N}$ is closed if the marked formulas occur in the path:*
(i) $k_1\ \ \varphi$ *and* $k_2\ \ \varphi$*, for any formula* φ *and* $k_1 \neq k_2$*;*

(ii) $n \quad \Box\varphi$ *or* $b \quad \Box\varphi$*, for any formula* $\Box\varphi$*;*
(iii) $n \quad \Diamond\varphi$ *or* $b \quad \Diamond\varphi$*, for any formula* $\Diamond\varphi$*.*

Definition 3.2 *A tableau of* $\mathcal{T}_{PM4N}$ *is closed if all of its branches are closed.*

The proofs and deductions in $\mathcal{T}_{PM4N}$ are as the usual. We denote that a formula φ is deduced from Γ in the tableau system $\mathcal{T}_{\mathbf{PM4N}}$ by $\Gamma \Vdash \varphi$. It may eventually happen that $\Gamma = \emptyset$ and in this case we will denote it by $\Vdash \varphi$, omitting Γ.

The deduction $\Gamma \Vdash \varphi$ is Tarskian, that is, (i) if $\varphi \in \Gamma$, then $\Gamma \Vdash \varphi$; (ii) if $\Gamma \Vdash \varphi$ and $\Gamma \subseteq \Sigma$, then $\Sigma \Vdash \varphi$; (iii) if $\Gamma \Vdash \varphi$ and $\Sigma \cup \{\varphi\} \Vdash \psi$, then $\Gamma \cup \Sigma \Vdash \psi$.

4 Galois pairs

In this section, we present some basic algebraic concepts and the pairs of functions motivated by the Galois connections, which will be used for further developments. For references on algebraic logic we indicate the books [7], [8], [9] and [10]; for elements of Galois pairs we suggest [11], [12], [13] and [14]. The book [7] also make developments on Galois connections.

Definition 4.1 *A binary relation* $\leq$ *on a set* A *is a partial order if it is reflexive, antisymmetric and transitive, that is, respectively:*
(i) *for all* $a \in A$*,* $a \leq a$*;*
(ii) *for all* $a, b \in A$*, if* $a \leq b$ *and* $b \leq a$*, then* $a = b$*;*
(iii) *for all* $a, b, c \in A$*, if* $a \leq b$ *and* $b \leq c$*, then* $a \leq c$*.*

Definition 4.2 *A partially ordered set (poset) is a pair* $\langle A, \leq \rangle$*, in which* A *is a non-empty set and* $\leq$ *is a partial order over* A*.*

Definition 4.3 *Let* $\langle A, \leq \rangle$ *be a poset and* $a, b \in A$*. The supremum of the pair* $\{a, b\}$*, in case there is one, is the element* $c \in A$ *such that:*
(i) $a \leq c$ *and* $b \leq c$*;*
(ii) *if* $a \leq d$ *and* $b \leq d$*, then* $c \leq d$*.*

Definition 4.4 *Let* $\langle A, \leq \rangle$ *be a poset and* $a, b \in A$*. The infimum of the pair* $\{a, b\}$*, in case there is one, is the element* $e \in A$ *such that:*
(i) $e \leq a$ *and* $e \leq b$*;*
(ii) *if* $f \leq a$ *and* $f \leq b$*, then* $f \leq e$*.*

Usually, we denote the supremum of $\{a, b\}$ by $\sup\{a, b\}$ or $a \vee b$, and the infimum of $\{a, b\}$ by $\inf\{a, b\}$ or $a \wedge b$. The supremum of $\{a, b\}$ is the least upper bound of $\{a, b\}$, the infimum of $\{a, b\}$ is the greatest lower bound of $\{a, b\}$.

Definition 4.5 *If* $\langle R, \leq \rangle$ *is a poset, for which given any* $a, b \in R$ *there are the* $\inf\{a, b\}$ *and the* $\sup\{a, b\}$*, we name lattice the algebraic structure* $\langle R, \wedge, \vee \rangle$*, in which:*

$$a \wedge b = \inf\{a, b\} \qquad \textit{and} \qquad a \vee b = \sup\{a, b\}.$$

Proposition 4.6 *If $\langle R, \leq \rangle$ is a lattice, then, for all $a, b, c \in R$, the following laws holds:*

R_1 $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ *and* $(a \vee b) \vee c = a \vee (b \vee c)$ *[associativity];*

R_2 $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$ *[commutativity];*

R_3 $(a \wedge b) \vee b = b$ *and* $(a \vee b) \wedge b = b$ *[absorption].*

Proof: This result is quoted in [10]. ■

Definition 4.7 *If $f : (A, \leq_A) \longrightarrow (P, \leq_P)$ is a function between two partially ordered sets, then:*

(i) *the function f preserves the orders, if $a \leq_A b$ implies $f(a) \leq_P f(b)$;*

(ii) *the function f inverts the orders, if $a \leq_A b$ implies $f(b) \leq_P f(a)$.*

In the context of mathematical analysis, usually these functions are called increasing and decreasing, respectively; but this denomination is less usual in the context of Galois connections. Other times they are called isotone and antitone, being that the first ones also occur with the name of monotone.

Definition 4.8 *Let $f : (A, \leq_A) \longrightarrow (A, \leq_A)$ be a function. Then:*

(i) *f is idempotent, if $f \circ f = f$;*

(ii) *f is extensive or inflationary if, for all $a \in A$, $a \leq f(a)$;*

(iii) *f is deflationary if, for all $a \in A$, $f(a) \leq a$.*

Definition 4.9 *Let $f : (A, \leq_A) \longrightarrow (A, \leq_A)$ be a function. Then:*

(i) *f is a Tarski operator (deductive closure operator) if is extensive (or inflationary), preserves orders and is idempotent;*

(ii) *f is an interior operator if is deflationary, preserves orders and is idempotent.*

When we analyse the definition of Galois connection, we will be able to make four simple permutations, what will generate us other pairs of functions, which maintain some similarity with the definition of connection.

As usual, the symbol $\Leftrightarrow$ must be understood as 'if, and only if'.

Definition 4.10 *If $(A, \leq_A)$ and $(P, \leq_P)$ are partially ordered sets, $a \in A$ and $p \in P$ are any elements and $f : A \longrightarrow P$ and $g : P \longrightarrow A$ are functions, then:*

(i) *the pair (f, g) is a Galois connection if $a \leq_A g(p) \Leftrightarrow p \leq_P f(a)$;*

(ii) *the pair $(f, g)^d$ is a dual Galois connection if $g(p) \leq_A a \Leftrightarrow f(a) \leq_P p$;*

(iii) *the pair $[f, g]$ is an adjunction if $a \leq_A g(p) \Leftrightarrow f(a) \leq_P p$;*

(iv) *the pair $[f, g]^d$ is a dual adjunction if $g(p) \leq_A a \Leftrightarrow p \leq_P f(a)$.*

The name adjunction comes from the theory of categories. In many texts on the subject, the pair $[f, g]$ is also called *residuated*.

If $(A, \leq_A)$ is a partially ordered set, then we denote the inverse order of $\leq_A$ by $\leq_A^{op}$ and, thereby, $(A, \leq_A^{op}) = (A, (\leq_A)^{-1})$. Thus:

$$a \leq_A b \Leftrightarrow b \leq_A^{op} a.$$

From this definition stem the following results.

Proposition 4.11 *Let $(A, \leq_A)$ and $(P, \leq_P)$ be partially ordered sets and $f : A \to P$ and $g : P \to A$ functions:*

(i) *if $[f, g]$ is an adjunction, then $[g, f]^d$ is a dual adjunction;*
(ii) *if $[f, g]^d$ is a dual adjunction, then $[g, f]$ is an adjunction;*
(iii) *if (f, g) is a Galois connection, then (g, f) is also a Galois connection;*
(iv) *if $(f, g)^d$ is a dual Galois connection, then $(g, f)^d$ is also a dual Galois connection.*

Proof: It is immediate. ■

Proposition 4.12 *If (f, g) is a Galois connection for $(A, \leq_A)$ and $(P, \leq_P)$, then:*

(i) *$(f, g)^d$ is a dual Galois connection for $(A, \leq_A^{op})$ and $(P, \leq_P^{op})$;*
(ii) *$[f, g]$ is an adjunction for $(A, \leq_A)$ and $(P, \leq_P^{op})$;*
(iii) *$[f, g]^d$ is a dual adjunction for $(A, \leq_A^{op})$ and $(P, \leq_P)$.*

Proof: It is immediate. ■

Following, we emphasize the adjunctions, as particular cases of Galois pairs. We enunciate various results, which, with the proper particularities, can be applied to the remaining Galois pairs.

In general, we do not indicate the orders $\leq_A$ and $\leq_P$, because the context allows for the identification of over which set we address of the order in question.

The following proposition gives us conditions to have an adjunction.

Proposition 4.13 *Let $(A, \leq_A)$ and $(P, \leq_P)$ be two partial orders, $f : A \to P$ and $g : P \to A$ functions, with $a, b \in A$ and $p, q \in P$. Then, the pair $[f, g]$ is an adjunction if, and only if, the following conditions hold:*

(i) *$a \leq g(f(a))$;*
(ii) *$f(g(p)) \leq p$;*
(iii) *$a \leq b \Rightarrow f(a) \leq f(b)$;*
(iv) *$p \leq q \Rightarrow g(p) \leq g(q)$.*

Proof: A proof can be met in [15]. ■

Thus, we have another way of defining adjunction: the pair $[f, g]$ is an adjunction if the functions f and g preserve the orders and the composites $g \circ f$ and $f \circ g$ are, respectively, inflationary and deflationary.

Proposition 4.14 *If the pair $[f, g]$ is an adjunction for the partial orders $(A, \leq_A)$ and $(P, \leq_P)$, then $f(a) = f(g(f(a)))$ e $g(b) = g(f(g(b)))$.*

Proof: A proof can be met in [15]. ■

Proposition 4.15 *If $[f, g]$ is an adjunction for $(A, \leq_A)$ and $(P, \leq_P)$, then the two compositions $g \circ f$ and $f \circ g$ are Tarski and interior operators, respectively, on A and P.*

Proof: See [16] for a proof. ■

Proposition 4.16 *If the pair $[f, g]$ is an adjunction for the lattices $(A, \wedge, \vee)$ and $(P, \wedge, \vee)$, then:*

(i) *$f(x \vee y) = f(x) \vee f(y)$;*
(ii) *$g(x \wedge y) = g(x) \wedge g(y)$.*

Proof: A proof can be met in [15]. ■

Proposition 4.17 *If $[f, g_1]$ and $[f, g_2]$ are adjunctions for $(A, \leq_A)$ and $(P, \leq_P)$, then $g_1 = g_2$. If $[f_1, g]$ and $[f_2, g]$ are adjunctions for $(A, \leq_A)$ e $(P, \leq_P)$, then $f_1 = f_2$.*

Proof: See [16] for a proof. ■

Proposition 4.18 *If $[f, g]$ is an adjunction for $(A, \leq_A)$ and $(P, \leq_P)$, then:*
(i) $a \in g(P) \Leftrightarrow g(f(a)) = a$;
(ii) $p \in f(A) \Leftrightarrow f(g(p)) = p$;
(iii) $f(A) = f(g(P))$;
(iv) $g(P) = g(f(A))$.
Proof: See [16] for a proof. ■

Thus, each point $a \in g(P)$ is a fixed point of the function $g \circ f$, and each point $p \in f(A)$ is a fixed point of the function $f \circ g$.

Proposition 4.19 *If $[f, g]$ is an adjunction for $(A, \leq_A)$ and $(P, \leq_P)$, then:*
(i) $f(a) = min\{p \in P : a \leq g(p)\}$;
(ii) $g(p) = max\{a \in A : f(a) \leq p\}$.
Proof: A proof can be met in [15]. ■

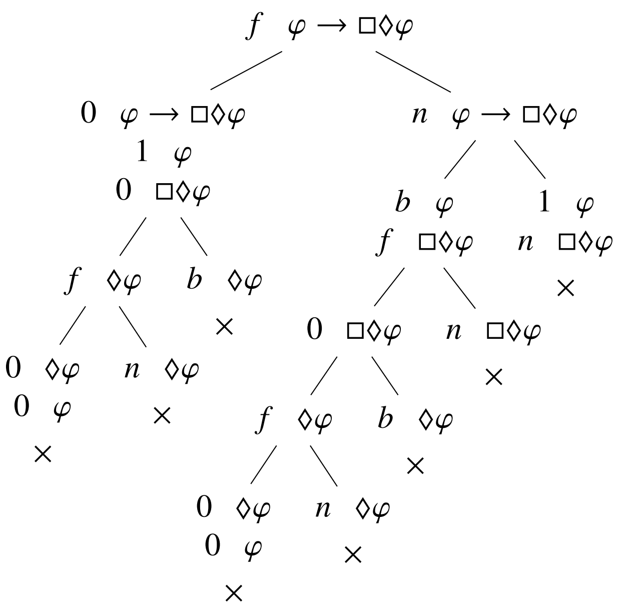
Proposition 4.20 *If $[f_1, g_1]$ is an adjunction for $(A, \leq_A)$ and $(B, \leq_B)$, and $[f_2, g_2]$ is an adjunction for $(B, \leq_B)$ and $(C, \leq_C)$, then $[f_2 \circ f_1, g_1 \circ g_2]$ is an adjunction for $(A, \leq_A)$ and $(C, \leq_C)$.*
Proof: See [16] for a proof. ■

Each Galois pair has similar results to the ones here enunciated on the adjunctions.
Following, we will see Galois pairs in PM4N.

5 The modal operators of PM4N as a Galois pair

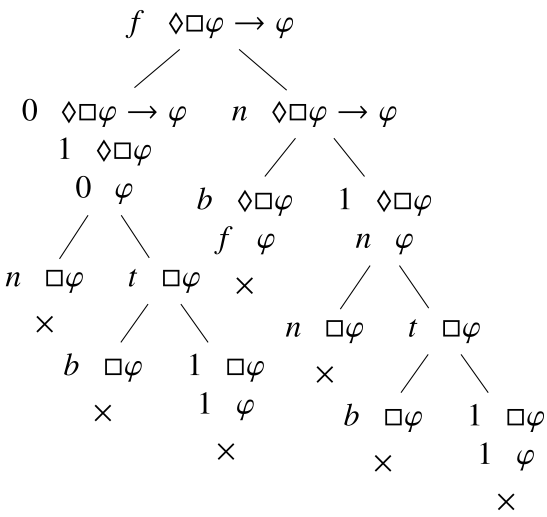
In this Section, we are going to consider the operators $\Diamond$ and $\Box$ as a Galois pair.
Now, we will use the tableaux system of Section 2.
Let's start by $(T_1) \Vdash \varphi \rightarrow \Box\Diamond\varphi$ and $(T_2) \Vdash \Diamond\Box\varphi \rightarrow \varphi$.

(T_1): $\Vdash \varphi \rightarrow \Box\Diamond\varphi$



DOI: 10.21167/cqdv25e25005 Disponível em: https://sistemas.fc.unesp.br/ojs/index.php/revistacqd/index

(T_2): $\Vdash \Diamond\Box\varphi \to \varphi$



Thus $\Diamond$ and $\Box$ are suitable to form a Galois pair, or more specifically, an adjunction. However, let us observe that the following conditions do not hold:

(T_3): $\varphi \to \psi \nVdash \Box\varphi \to \Box\psi$

For $v(\varphi) = 1$ and $v(\psi) = b$, we have $1 \to b$ and $1 \to 0$, then $v(\varphi \to \psi) = b$ and $v(\Box\varphi \to \Box\psi) = 0$.

(T_4): $\varphi \to \psi \nVdash \Diamond\varphi \to \Diamond\psi$

For $v(\varphi) = n$ and $v(\psi) = 0$, we have $n \to 0$ and $1 \to 0$, then $v(\varphi \to \psi) = b$ and $v(\Diamond\varphi \to \Diamond\psi) = 0$.

We do not had success in finding a Galois pair represented by the operators $\Diamond$ and $\Box$. But we can still refine this question. For this, the strict modal implication is going to be defined.

Definition 5.1 *Strict implication:* $\varphi \supset \psi := \Box(\varphi \to \psi)$.

The definition above generates the following table for the strict implication:

$\supset$	0	n	b	1
0	1	1	1	1
n	0	1	0	1
b	0	0	1	1
1	0	0	0	1

Proposition 5.2 *The strict implication $\supset$ establishes a partial order on the set of formulas of PM4N.*
Proof: (i) *We can observe that $\varphi \supset \varphi$ just looking for the principal diagonal of the table for $\supset$.*

(ii) *If $\varphi \supset \psi$ and $\psi \supset \varphi$, then $v(\varphi) = v(\psi)$, for if $v(\varphi) \neq v(\psi)$, then $\varphi \not\supset \psi$ or $\psi \not\supset \varphi$ or both when $v(\varphi) = k$, $v(\psi) = k'$ and $k \in \{n, b\}$.*

(iii) *If* $\varphi \supset \psi$ *and* $\psi \supset \sigma$*, then* $\varphi \supset \sigma$*. Let's consider that* $\varphi \supset \psi$ *and* $\psi \supset \sigma$*:*

- if $v(\varphi) = 0$*, then* $\varphi \supset \sigma$*.*

- if $v(\varphi) = n$*, as* $\varphi \supset \psi$*, then* $v(\psi) = n$ *or* $v(\psi) = 1$*. As* $\psi \supset \sigma$*, then then* $v(\sigma) = n$ *or* $v(\sigma) = 1$*;* $v(\sigma) = 1$*, respectively. In any case* $\varphi \supset \sigma$*.*

- if $v(\varphi) = b$*, as* $\varphi \supset \psi$*, then* $v(\psi) = b$ *or* $v(\psi) = 1$*. As* $\psi \supset \sigma$*, then then* $v(\sigma) = b$ *or* $v(\sigma) = 1$*;* $v(\sigma) = 1$*, respectively. Again* $\varphi \supset \sigma$*.*

- if $v(\varphi) = 1$*, by the hypotheses,* $v(\psi) = 1 = v(\sigma)$ *and* $\varphi \supset \sigma$*.* ∎

Now, we introduce specifics derived rules for the strict implication:

$$\begin{array}{llcll}
0 & & \varphi \supset \psi & & \\
\hline
r\ \varphi & \text{if } s < r & | & r\ \varphi & \text{if } r, s \text{ are incomparable} \\
s\ \psi & & | & s\ \psi &
\end{array}$$

$$\begin{array}{ccccccc}
1 & & & \varphi \supset \psi & & & \\
\hline
0\ \varphi & | & n\ \varphi & | & b\ \varphi & | & 1\ \psi \\
 & | & n\ \psi & | & b\ \psi & | &
\end{array}$$

The following result (T_5) is important, since we are going to be able to transfer some outcomes from the new implication to the old one with it.

(T_5) If $\Vdash \varphi \supset \psi$, then $\Vdash \varphi \to \psi$

$$\begin{array}{cccc}
\multicolumn{4}{c}{1\ \ \varphi \supset \psi} \\
\multicolumn{4}{c}{f\ \ \varphi \to \psi} \\
0\ \varphi & n\ \varphi & b\ \varphi & 1\ \psi \\
(i) & n\ \psi & b\ \psi & (iv) \\
 & (ii) & (iii) &
\end{array}$$

$$\begin{array}{ccc}
\multicolumn{3}{c}{(i)} \\
\multicolumn{3}{c}{0\ \ \varphi} \\
0\ \varphi \to \psi & \multicolumn{2}{c}{n\ \varphi \to \psi} \\
1\ \varphi & b\ \varphi & 1\ \varphi \\
\times & \times & \times
\end{array}
\qquad\qquad
\begin{array}{ccc}
\multicolumn{3}{c}{(ii)} \\
\multicolumn{3}{c}{n\ \ \varphi} \\
\multicolumn{3}{c}{n\ \ \psi} \\
0\ \varphi \to \psi & \multicolumn{2}{c}{n\ \varphi \to \psi} \\
1\ \varphi & b\ \varphi & 1\ \varphi \\
\times & \times & \times
\end{array}$$

$$\begin{array}{c}
(iii)\\
b\ \ \varphi\\
b\ \ \psi\\
\diagup \qquad \diagdown\\
\begin{array}{cc}
\begin{array}{c} 0\ \ \varphi\to\psi\\ 1\ \ \varphi\\ \times \end{array} &
\begin{array}{c} n\ \ \varphi\to\psi\\ \diagup \quad \diagdown\\ \begin{array}{cc} \begin{array}{c} b\ \ \varphi\\ f\ \ \psi\\ \times \end{array} & \begin{array}{c} 1\ \ \varphi\\ \times \end{array} \end{array} \end{array}
\end{array}
\end{array}
\qquad\qquad
\begin{array}{c}
(iv)\\
1\ \ \psi\\
\diagup \qquad \diagdown\\
\begin{array}{cc}
\begin{array}{c} 0\ \ \varphi\to\psi\\ 1\ \ \varphi\\ 0\ \ \psi\\ \times \end{array} &
\begin{array}{c} n\ \ \varphi\to\psi\\ \diagup \quad \diagdown\\ \begin{array}{cc} \begin{array}{c} b\ \ \varphi\\ f\ \ \psi\\ \times \end{array} & \begin{array}{c} 1\ \ \varphi\\ n\ \ \psi\\ \times \end{array} \end{array} \end{array}
\end{array}
\end{array}$$

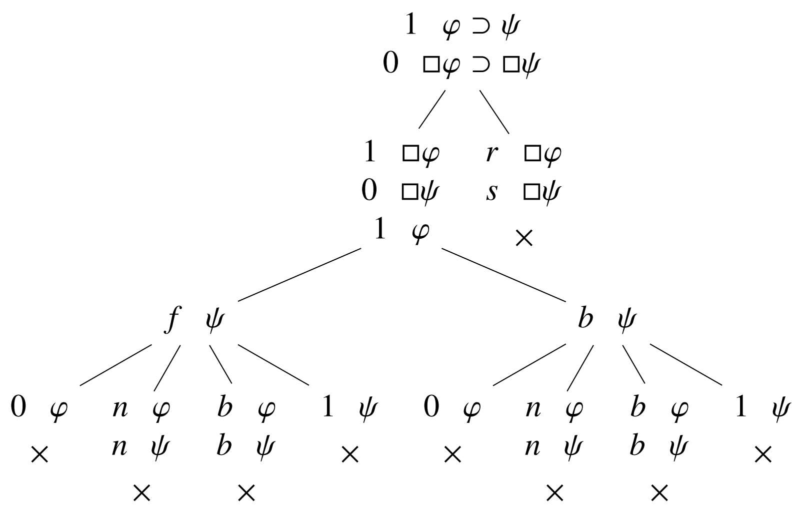
With the new implication, and also new order, we can obtain the results that if we take $f = \lozenge$ and $g = \square$, then the pair $[f, g] = [\lozenge, \square]$ is an adjunction, a particular Galois pair. Let us see:

(T_6) If $\Vdash \varphi \supset \psi$, then $\Vdash \lozenge\varphi \supset \lozenge\psi$

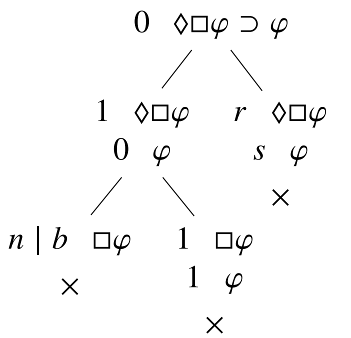
$$\begin{array}{c}
1\ \ \varphi\supset\psi\\
0\ \ \lozenge\varphi\supset\lozenge\psi\\
\diagup \qquad \diagdown\\
\begin{array}{cc}
\begin{array}{c}
1\ \ \lozenge\varphi\\ 0\ \ \lozenge\psi\\ 0\ \ \psi\\
\diagup \qquad\qquad \diagdown\\
\begin{array}{cc}
\begin{array}{c} n\ \ \varphi\\
\begin{array}{cccc}
\begin{array}{c} 0\ \ \varphi\\ \times \end{array} &
\begin{array}{c} n\ \ \varphi\\ n\ \ \psi\\ \times \end{array} &
\begin{array}{c} b\ \ \varphi\\ b\ \ \psi\\ \times \end{array} &
\begin{array}{c} 1\ \ \psi\\ \times \end{array}
\end{array}
\end{array} &
\begin{array}{c} t\ \ \varphi\\
\begin{array}{cccc}
\begin{array}{c} 0\ \ \varphi\\ \times \end{array} &
\begin{array}{c} n\ \ \varphi\\ n\ \ \psi\\ \times \end{array} &
\begin{array}{c} b\ \ \varphi\\ b\ \ \psi\\ \times \end{array} &
\begin{array}{c} 1\ \ \psi\\ \times \end{array}
\end{array}
\end{array}
\end{array}
\end{array} &
\begin{array}{c} r\ \ \lozenge\varphi\\ s\ \ \lozenge\psi\\ \times \end{array}
\end{array}
\end{array}$$

When we apply a rule in which r, s are incomparable, we will take $r, s \in \{n, b\}$. And the tableau for a $\supset$-formula will always begin with the formula taking a valuation 0 or 1, since $\supset$ does not admit non-classical values n and b.

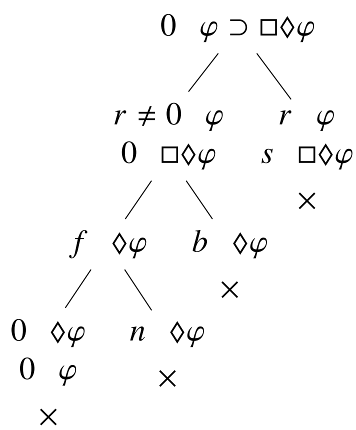
(T_7) If $\Vdash \varphi \supset \psi$, then $\Vdash \square\varphi \supset \square\psi$



$(T_8) \Vdash \Diamond\Box\varphi \supset \varphi$



$(T_9) \Vdash \varphi \supset \Box\Diamond\varphi$



Therefore, we now have a Galois adjunction determined by $f = \Diamond$ and $g = \Box$.

By defining a new bi-implication $\varphi \rightleftarrows \psi$ with the strict implication, the results on adjunction may be replicated here.

Definition 5.3 *Strict bi-implication:* $\varphi \rightleftarrows \psi := (\varphi \supset \psi) \wedge (\psi \supset \varphi)$.

The definition above generates the following table for the strict bi-implication:

$\rightleftarrows$	0	n	b	1
0	1	0	0	0
n	0	1	0	0
b	0	0	1	0
1	0	0	0	1

Proposition 5.4 *The following results are all valid for any* $\varphi, \psi \in For(PM4N)$*:*

$(T_{10}) \Vdash \varphi \supset \Box\psi \Leftrightarrow \Vdash \Diamond\varphi \supset \psi$;

$(T_{11}) \Vdash \Diamond(\varphi \vee \psi) \rightleftarrows (\Diamond\varphi \vee \Diamond\psi)$;

$(T_{12}) \Vdash \Box(\varphi \wedge \psi) \rightleftarrows (\Box\varphi \wedge \Box\psi)$;

$(T_{13}) \Vdash \Diamond\varphi \rightleftarrows \Diamond\Box\Diamond\varphi$;

$(T_{14}) \Vdash \Box\varphi \rightleftarrows \Box\Diamond\Box\varphi$.

Proof: It is straightforward. Comes from the fact that $[\Diamond, \Box]$ *is an adjunction with the strict implication.* ■

Proposition 5.5 *The following results are all valid for any* $\varphi, \psi \in For(PM4N)$*:*

$(T_{15}) \Vdash \Diamond(\varphi \vee \psi) \leftrightarrow (\Diamond\varphi \vee \Diamond\psi)$;

$(T_{16}) \Vdash \Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$;

$(T_{17}) \Vdash \Diamond\varphi \leftrightarrow \Diamond\Box\Diamond\varphi$;

$(T_{18}) \Vdash \Box\varphi \leftrightarrow \Box\Diamond\Box\varphi$.

Proof: Follows from (T_5) *and Proposition 5.4.* ■

6 Final considerations

The logic PM4N has a plea from algebra more specifically the partial order over its four values, but it has been developed as a many-valued logic with interesting relations with modal operators and modal logics.

We would like to use to use more mathematical tools for to share more laws or results about the logic PM4N.

In this paper we have presented some elements of the Galois pairs, as functions defined between partial orders. And using the theoretical elements we have obtained another implication for PM4N and with this new implication plus the Galois notions we got some new results for PM4N.

So we have developed an interesting interaction between logic and algebra that can be applied to other logics and also other structures.

7 Acknowledgements

This work was realized with the financial support of the Coordination for the Improvement of Higher Education Personnel - Brazil (CAPES).

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