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Fractional Logistic Equation Applied to Brazilian COVID-19 Data

Abstract

In this paper we analyzed a fractional logistic model applied to Brazilian COVID-19 data. In order to solve the logistic model, we presented a modified version of Adams-Bashforth numerical method. All numerical simulations were performed in MATLAB. As a result, the qualitative comparison between the theoretical model and the cumulative confirmed cases showed a great proximity. In this case, the coefficient of determination, R^2, between these data sets, was calculated to be $0,917$, attesting that the fractional model can be used to model real data with quality.

Keywords: Fractional Calculus. Caputo Fractional Derivative. Adams-Bashforth method. Fractional Logistic Equation. COVID-19.

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1 Introduction

In December 2019 in the city Wuhan in China the first case of a new respiratory syndrome that came to be called coronavirus disease 2019, or just COVID-19, was reported. This disease caused by the new coronavirus SARS-COV-2 quickly spread and within months had already reached every continent on the planet (TENE et al., 2021). The disease was declared a global pandemic by the WHO (World Health Organization) on March 11, 2020 in a scenario with more than 110,000 cases distributed in 114 countries (CAVALCANTE et al., 2020).

The first cases were confirmed in Brazil in February 2020, and since then several actions have been implemented to reduce the impact of the disease in the country. In the beginning, data have been collected and made available by the Brazilian Ministry of Health, which allowed us to understand the dynamics and progression of the disease. However, all the measures implemented were not enough to mitigate the impacts of COVID-19 and the urgency to find a vaccine grew (CAVALCANTE et al., 2020).

While this urgency was high on the agendas of governments around the world, the scientific community has been using this crisis to map vulnerabilities by modeling reported outbreak data from around the world to predict and prevent the occurrence of similar events. Thus, several mathematical approaches have been proposed (TENE et al., 2021; SHEN, 2020) to accurately estimate the infection growth rate, tipping point, duration of outbreaks, and total number of cases. Among these are simple logistic function-based models, improved SIR and SEIR-like models, hierarchical polynomial regression models, and others (TENE et al., 2021).

Several studies have been conducted, such that of Pelinovsky *et al.* (PELINOVSKY et al., 2020) who applied a simple mathematical approach based on the logistic equation to study the outbreaks of COVID-19 in different countries taking into account the first months of the pandemic. His proposed logistic model was then considered adequate for most of the countries studied and allowed the estimation of the growth rate and the expected number of infected. Wang *et al* (WANG et al., 2020) also conducted a similar study and predicted the epidemic trends of the disease using the logistic model and the machine learning time series prediction model (TENE et al., 2021). Through all these works and several others that can be easily found in research bases, one can see that the logistic model is a great tool to explore the behavior of COVID-19 here in Brazil.

In this sense, this paper aim to present a model using a fractional order version of the logistic equation for COVID-19 dynamics. The fractional calculus will be applied here in order to verify its applicability to the context of the evolution of the disease. It has been applied in several areas of engineering, physics, finance, applied mathematics, bioengineering, among others (GUO; XU, 2006; FREED; DIETHELM, 2006).

The fractional calculus appeared in the 17th century, when L'Hôspital sent a letter to Libeniz asking about the possibility of a derivative of order 1/2 (ROSS, 1977). And it was the answer to this letter that culminated in the first definitions of derivative and integral of non-integer order. Laplace in 1812, presented the first definition of fractional derivative which gave room for others to investigate the consequences of these definitions. Thus, the fractional calculus was studied and improved until in 1969 Caputo proposed the fractional derivative operator that is now applied more abundantly to solve physical problems (CAMARGO; OLIVEIRA, 2015; VARALTA; GOMES; CAMARGO, 2014).

In this sense, this study aims to apply the fractional logistic equation to evaluate the dynamics of COVID-19 accumulated cases in Brazil in the period ranging from february 26, 2020, to october 20, 2021. We intend to assess the effectiveness of the fractional model on studying the infection rate and the total number of cases during the given period. That way, the presented manuscript is divided as follows: in section 2 we present the definition of Caputo fractional derivative, section 3 presents the fractional version of the logistic equation, and in section 4 the numerical method used to solve the fractional differential equations, called modified Adams-Bashforth method, is presented. Finally, in section 5 the application of the fractional logistic equation to describe the cumulative confirmed cases of COVID-19 in Brazil during the first outbreak is shown.

2 Caputo Fractional Derivative

In this section we present the definiton of the fractional differential operator that will be used in this text. Specifically, we work with Caputo's fractional derivative because their physical interpretations. This definition is based on the following reference (CAMARGO; OLIVEIRA, 2015), in which all details about this operator can be found.

The fractional Caputo derivative of order α is defined by (CAMARGO; OLIVEIRA, 2015)

$$ {}_{b}^{C}\mathbf{D}_{t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{b}^{t} \frac{f^{(n)}(u)}{(t-u)^{\alpha-n+1}} du, \tag{1} $$

where $\alpha \in \mathbb{R}_{>0}$, $b \in \mathbb{R}$, $n = \lceil \alpha \rceil$, $f^{(n)}(u) = d^n f(u)/du^n$ is the nth integer order derivative and $\Gamma(\cdot)$ is the Gamma function. Caputo derivatives can be defined for $\alpha \in \mathbb{C}$, when $\Re(\alpha) > 0$, but, here will be considered only $\alpha \in \mathbb{R}_{>0}$. From now on, since there is no confusion with another definition of fractional derivative, we will write the derivative operator in the following form ${}_{0}^{C}\mathbf{D}_{t}^{\alpha} = \mathbf{D}_{t}^{\alpha}$. Note that the Caputo fractional derivative of a constant function vanishes, i.e. $\mathbf{D}_{t}^{\alpha} C = 0$, when C is a constant. This fact allows us to continue using the initial conditions as in the classical differential equations case what is useful in when considering applications like the one treated here.

3 Fractional Logistic Equation

In this section we present the fractional version of the logistic equation. The standard logistic ordinary differential equation is a simplified model that can be used to model several real systems (TENE et al., 2021; WANG et al., 2020). We can cite, for example, the standard population growth model due to Verhulst (VERHULST, 1838) and also an application in medicine where the logistic differential equation is used to model the growth of tumors (VARALTA; GOMES; CAMARGO, 2014).

The fractional version of the dimensionless logistic equation is a straightforward modification of the standard dimensionless logistic equation and is given by

$$ \mathbf{D}_{t}^{\alpha} y(t) = \rho y(t)(1 - y(t)), \tag{2} $$

where $t > 0$, $\rho > 0$ is a constant and $0 < \alpha \leq 1$. Notice that when $\alpha = 1$, Eq.(2) recovers the standard and well known dimensionless logistic model

$$ \frac{dy}{dt} = \rho y(t)(1 - y(t)), \tag{3} $$

which has its exact solution given by

$$y(t) = \frac{y(0)}{[1 - y(0)]e^{-\rho t} + y(0)}. \tag{4}$$

It is immediate to note that both the fractional model and standard model are nonlinear equations. For the standard model, one can solve directly to obtain Eq.(4) since that ODE is separable. On the other hand, an analytical solution for the fractional logistic equation can not be obtained through the same methods.

A linearized version of the fractional logistic equation,

$$\mathbf{D}_t^\alpha w(t) = \rho(1 - w(t)), \tag{5}$$

where $w(t) = 1/y(t)$, was treated and solved analytically using the Laplace transform in (VARALTA; GOMES; CAMARGO, 2014). The solution found by the authors is given in terms of Mittag-Leffler function as

$$y(t) = \frac{y(0)}{[1 - y(0)]E_\alpha(-\rho t^\alpha) + y(0)} \tag{6}$$

and yields a nice approximation when α is close to 1, since it resembles the standard form analytical solution Eq.(4). But, can be shown that the linearized equation solution, Eq.(6), fails to provide a good estimation to the fractional logistic equation, Eq.(2), as α is far from the unity. Therefore, a reliable numerical method will be needed to find the approximated solution to the problem.

4 Numerical Solution of the Fractional Logistic Equation

In this section we present a method that can be used to solve the fractional logistic equation numerically: the modified Adams-Bashforth method. The content discussed in here is based in the reference (DIETHELM; FORD; FREED, 2002). This method was proposed by Diethelm et al. (DIETHELM; FORD; FREED, 2002; DIETHELM, 2010) and it is a predictor-corrector scheme. Further details, as error analysis and convergence, can be founded in the following references (DIETHELM; FORD; FREED, 2002; DIETHELM et al., 2005).

First, consider the initial value problem given by the fractional differential equation

$$\mathbf{D}_t^\alpha y(t) = f(t, y(t)), \tag{7}$$

where $\alpha \in (0, 1)$, $t \in [0, T]$, with initial condition $y(0) = y_0$.

Now, lets define the integral operator

$$\mathbf{J}_t^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \omega)^{\alpha - 1} g(\omega) d\omega, \tag{8}$$

which is known as Riemann-Liouville fractional integral (DIETHELM; FORD; FREED, 2002; OWOLABI; ATANGANA, 2019).

Our next goal is to understand how the Riemann-Lioville fractional integral operator $\mathbf{J}_t^\alpha$ acts on the Caputo fractional derivative operator $\mathbf{D}_t^\alpha$, defined by the Eq.(1). Given a sufficient smooth function $z(t)$, then we apply this operator $\mathbf{J}_t^\alpha$ in $\mathbf{D}_t^\alpha z(t)$ to obtain

$$\begin{aligned}
\mathbf{J}_t^\alpha \mathbf{D}_t^\alpha z(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t - \omega)^{\alpha - 1} \mathbf{D}_\omega^\alpha z(\omega) d\omega \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t - \omega)^{\alpha - 1} \left[\frac{1}{\Gamma(1 - \alpha)} \int_0^\omega (\omega - \mu)^{-\alpha} z'(\mu) d\mu \right] d\omega \\
&= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - \omega)^{\alpha - 1} \left[\int_0^\omega (\omega - \mu)^{-\alpha} z'(\mu) d\mu \right] d\omega. \qquad (9)
\end{aligned}$$

Performing a change in the integration order we can write Eq.(9) as

$$\mathbf{J}_t^\alpha \mathbf{D}_t^\alpha z(t) = \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(1-\alpha)} \int_0^t z'(\mu) \left[\int_\mu^t (t-\omega)^{\alpha-1} (\omega-\mu)^{-\alpha} d\omega \right] d\mu. \tag{10}$$

Taking $s = \omega - \mu$, Eq.(10) becomes

$$\begin{aligned} \mathbf{J}_t^\alpha \mathbf{D}_t^\alpha z(t) &= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(1-\alpha)} \int_0^t z'(\mu) \left[\int_0^{t-\mu} s^{-\alpha} (t-\mu-s)^{\alpha-1} ds \right] d\mu \\ &= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(1-\alpha)} \int_0^t z'(\mu) \left[\int_0^{t-\mu} s^{-\alpha} (t-\mu)^{\alpha-1} \left(1 - \frac{s}{t-\mu}\right)^{\alpha-1} ds \right] d\mu. \end{aligned} \tag{11}$$

Taking now $r = t/(t-\mu)$, then the integral in Eq.(11) can be written in the form

$$\mathbf{J}_t^\alpha \mathbf{D}_t^\alpha z(t) = \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(1-\alpha)} \int_0^t z'(\mu) \left[\int_0^1 r^{(1-\alpha)-1} (1-r)^{\alpha-1} dr \right] d\mu. \tag{12}$$

It can be shown that the inner integral in Eq.(12) is (CAMARGO; OLIVEIRA, 2015)

$$\int_0^1 r^{(1-\alpha)-1} (1-r)^{\alpha-1} dr = \Gamma(\alpha)\Gamma(1-\alpha) \tag{13}$$

and consequently, the operator $\mathbf{J}_t^\alpha$ acts in Caputo's fractional derivative $\mathbf{D}_t^\alpha z(t)$ as

$$\mathbf{J}_t^\alpha \mathbf{D}_t^\alpha z(t) = \int_0^t z'(\mu) d\mu = z(t) - z(0). \tag{14}$$

Finally, we can apply the operator $\mathbf{J}_t^\alpha$ in Eq.(7) to find that

$$y(t) = y(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\omega)^{\alpha-1} g(\omega) d\omega, \tag{15}$$

where $g(\omega) = f(\omega, y(\omega))$. This last equation is a key ingredient in the procedure that will be presented next.

Consider $\{0 = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = T\}$ a discretization of $[0, T]$ with $t_k - t_{k-1} = h$, for all $k \in \{1, 2, 3, \ldots, n+1\}$. Then, at the point $t = t_{k+1}$ we have

$$\begin{aligned} y_{k+1} &= y_0 + \frac{1}{\Gamma(\alpha)} \int_0^{t_{k+1}} (t_{k+1} - \omega)^{\alpha-1} g(\omega) d\omega \\ &= y_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} (t_{k+1} - \omega)^{\alpha-1} g(\omega) d\omega, \end{aligned} \tag{16}$$

where $y(t_k) = y_k$, for all $k \in \{0, 1, 2, \ldots, n+1\}$. Now we use the trapeizodal rule in Eq.(16) considering $(t_{k+1} - \cdot)^{\alpha-1}$ as a weight function. Then, if $g_{j+1}(\omega)$ is a linear interpolation of function $g(\omega)$ in the interval $[t_j, t_{j+1}]$ we get

$$g_{j+1}(\omega) = \frac{\omega - t_j}{h} g(t_{j+1}) - \frac{\omega - t_{j+1}}{h} g(t_j). \tag{17}$$

So, the integrals in the right side of Eq.(16) can be estimated by the following

$$\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}}(t_{k+1}-\omega)^{\alpha-1}g(\omega)d\omega \approx \sum_{j=0}^{k}\int_{t_j}^{t_{j+1}}(t_{k+1}-\omega)^{\alpha-1}g_{j+1}(\omega)d\omega$$
$$= \sum_{j=0}^{k+1}\varphi_{j,k+1}g(t_j), \tag{18}$$

in which the coefficients $\varphi_{j,k+1}$ can be calculated by

$$\varphi_{0,k+1} = -\frac{1}{h}\int_{t_0}^{t_1}(t_{k+1}-\omega)^{\alpha-1}(\omega-t_1)\,d\omega, \tag{19}$$

$$\varphi_{j,k+1} = \frac{1}{h}\int_{t_{j-1}}^{t_j}(t_{k+1}-\omega)^{\alpha-1}\left(\omega-t_{j-1}\right)d\omega$$
$$-\frac{1}{h}\int_{t_j}^{t_{j+1}}(t_{k+1}-\omega)^{\alpha-1}\left(\omega-t_{j+1}\right)d\omega,\ \ j\in\{1,\dots,k\} \tag{20}$$

$$\varphi_{k+1,k+1} = \frac{1}{h}\int_{t_k}^{t_{k+1}}(t_{k+1}-\omega)^{\alpha-1}(\omega-t_k)\,d\omega. \tag{21}$$

We notice that the integrals Eqs.(19)–(21) can be easily calculated. Their result can be found through the change of variables $\zeta = t_{k+1}-\omega$ and the use of the indefinite integral $\int \zeta^{\beta}d\zeta = \zeta^{\beta+1}/(\beta+1)+\text{constant}$, when $\beta \neq -1$. After all the calculations, we obtain the following coefficients $\varphi_{j,k+1}$ as

$$\varphi_{0,k+1} = \frac{h^{\alpha}}{\alpha(\alpha+1)}\left[k^{\alpha+1}-(k-\alpha)(k+1)^{\alpha}\right], \tag{22}$$

$$\varphi_{j,k+1} = \frac{h^{\alpha}}{\alpha(\alpha+1)}\left[(k-j+2)^{\alpha+1}+(k-j)^{\alpha+1}\right.$$
$$\left. - 2(k-j+1)^{\alpha+1}\right],\ \ j\in\{1,\dots,k\}, \tag{23}$$

$$\varphi_{k+1,k+1} = \frac{h^{\alpha}}{\alpha(\alpha+1)}. \tag{24}$$

Finally, we can write the approximate solution in a instant $t = t_{k+1}$ which is given by

$$y_{k+1} = y_0 + \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{k}\varphi_{j,k+1}f(t_j,y_j) + \frac{1}{\Gamma(\alpha)}\varphi_{k+1,k+1}f(t_{k+1},y_{k+1}^{p}). \tag{25}$$

Note that in the formula given by Eq.(25), which approximates the differential equation solution, the value y_{k+1}, which represents the function $y(t)$ evaluated at the instant t_{k+1}, appears naturally on the right side of the expression as an argument of f. This is a problem since this value is not known, in fact y_{k+1} is what we want to calculate. Since it is an argument of f we need to, at least, estimate it and the estimation will be called y_{k+1}^{p}. To do so, we will use a procedure similar to the one described above, modifying only the quadrature performed on the integrals of Eq.(16), instead of approximating them using the trapezoidal rule, we will consider the function within each subinterval

$[t_j, t_{j+1}]$ as a constant and equal to $g(t_j)$. That is,

$$\sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} (t_{k+1} - \omega)^{\alpha-1} g(\omega) d\omega \approx \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} (t_{k+1} - \omega)^{\alpha-1} q_{j+1}(\omega) d\omega$$
$$= \sum_{j=0}^{k} \varrho_{j,k+1} q(t_j), \tag{26}$$

where $q_{j+1}(\omega) = g(t_j)$, for all $\omega \in [t_j, t_{j+1}]$. In this way the coefficients $\varrho_{j,k+1}$ are

$$\begin{aligned} \varrho_{j,k+1} &= \int_{t_j}^{t_{j+1}} (t_{k+1} - \omega)^{\alpha-1} d\omega \\ &= \frac{h^\alpha}{\alpha} \left[(k - j + 1)^\alpha - (k - j)^\alpha \right]. \end{aligned} \tag{27}$$

Finally, from Eq.(16), y^p_{k+1} can be written as

$$y^p_{k+1} = y_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k} \varrho_{j,k+1} f(t_j, y_j). \tag{28}$$

Thus, the numerical solution obtained by Adam-Bashforth method is constructed in two steps. First, we calculate the predictor y^p_{k+1}, using Eq.(28), and then the numerical solution is adjusted through the corrector given by Eq.(25). This gives us the desired approximated solution y_{k+1}, that corresponds to $y(t_{k+1})$. In the next section we will apply this numerical method in the analysis of COVID data.

5 Application to COVID-19 Data in Brazil

In this section the fractional logistic equation will be applied to describe the cumulative confirmed cases of COVID-19 from Brazil during first outbreak.

Historically, the first confirmed case of COVID-19 in Brazil was on February 26, 2020. Up to this day, the number of confirmed cases, suspect cases, negative cases and deaths caused throughout the pandemic has been constantly updated through Brazilian Government official websites, in particular on the Ministry of Health page. These data can also be retrieved from a number of reliable public available online repositories. In this perspective, in order to describe the official data through the fractional growth model we considered data from February 26, 2020, to October 20, 2021, publicly available on the website ⟨https://ourworldindata.org/coronavirus/country/brazil⟩Our World in Data, a project of the ⟨https://global-change-data-lab.org/⟩Global Change Data Lab, a registered charity in England and Wales (ROSER et al., 2020).

Numerical simulation was performed in MATLAB for the following initial value problem

$$\mathbf{D}^\alpha_t y(t) = \rho y(t) \left(1 - \frac{y(t)}{y_\infty} \right), \tag{29}$$

where the initial data is $y(0) = 1$, the growth rate is $\rho = 0,0248$, the carrying capacity is $y_\infty = 2,5 \times 10^7$, the fractional derivative parameter $\alpha = 0,40$ and $t \in [0, 602]$.

Figure 1 shows a comparison between the fractional growth model, determined by the described numerical approach, and the corresponding Brazilian COVID-19 publicly available data. A qualita-

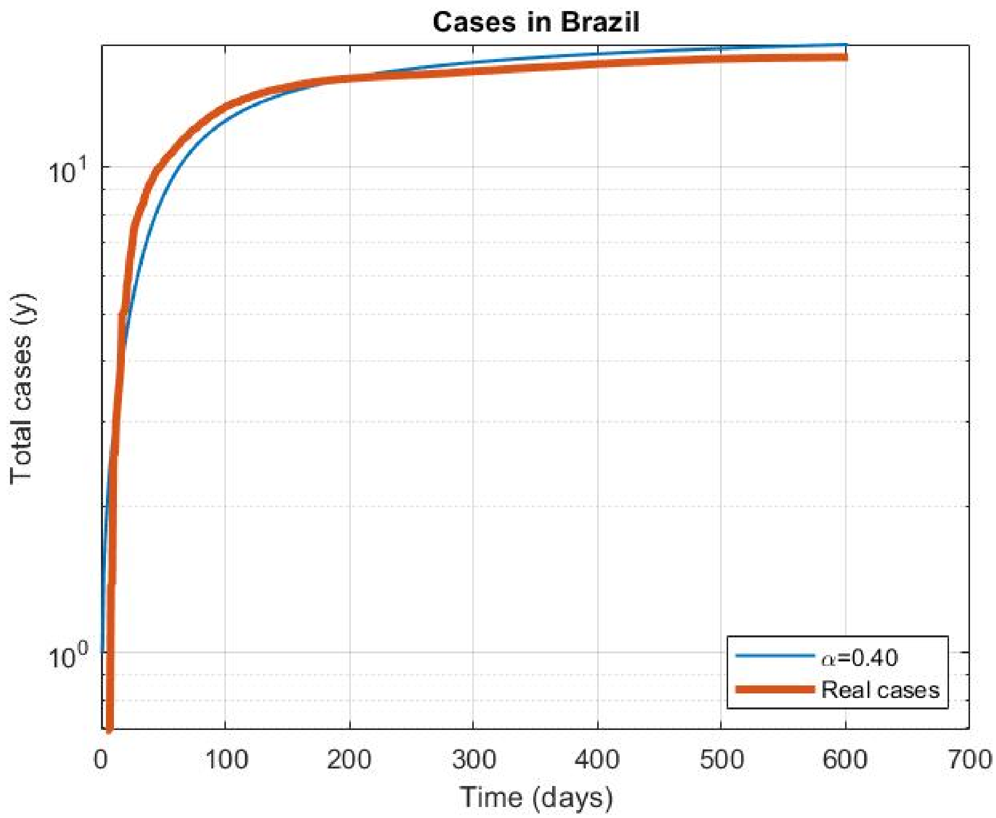

Figure 1: Fractional Growth Model Fit compared to confirmed COVID-19 data from Brazil

tive comparison between the model and the cumulative confirmed cases on a semilog scale shows great proximity between them. In order to measure quantitatively this similarity, the coefficient of determination was used. This coefficient can be employed to understand how well the fractional model fits the data set. It can be calculated as

$$R^2 = 1 - \frac{SS_{\text{res}}}{SS_{\text{tot}}}, \tag{30}$$

where $SS_{\text{res}} = \sum_j (y_j - y(t_j))^2$ is the residual sum of squares and $SS_{\text{tot}} = \sum_j (y_j - \bar{y})^2$ is the total sum of squares. Here, y_j is the number of accumulated cases on the data set at day t_j, $y(t_j)$ is the accumulated number of cases given by the fractional model at t_j and $\bar{y}$ is the average of all accumulated cases. Finally, the coefficient of determination R^2 calculated for this model is equals to $0,917$, what can be considered as an excellent fit by the model considered.

6 Conclusions

In this work we analyzed the fractional logistic equation as a tool to understand Brazilian COVID-19 public data behaviour. For this propose, we used the modified Adams-Bashforth numerical method to solve the logistic model. In this sense, we compared the numerical simulations with real data to obtain a close fit between the fractional logistic model and the real Brazilian COVID-19 data. This fact is corroborated by the coefficient of determination R^2 equals to $0,917$. In the sequence of this study we can improve the theoretical model using, for example, the generalized logistic equation, Gompertz equation or SEIR-like models. In addition, we can use Monte-Carlo method to estimate some parameters related to subject such as the parameter ρ given in the present work.

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