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A new tableaux system for KD

Um novo sistema de tablôs para KD

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Resumo

Lógica deôntica é um caso particular de lógica modal, que trata de relacões entre os conceitos de obrigação, permissão e proibição. Para este artigo, consideramos a lógica deôntica básica, denotada por **KD**, para a qual introduzimos um sistema de tablôs distinto dos usuais sistemas de Kripke de **KD**.

Palavras-chave: Lógicas modais. Lógica deôntica. Sistemas dedutivo. Tablôs. Árvores de refutação

Abstract

The deontic logic is a particular case of modal logic that analyses relations among the concepts of obligation, permission and prohibition. We take the standard deontic logic **KD**, for which we introduce a new system of tableaux.

Keywords: Modal logics. Deontic Logic. Deduction systems. Tableaux. Refutation Trees.

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1 Introduction

Modal logics appeared in the early 20th century, to formalize alethic terms of necessity and possibility. The first papers formalized expressions like 'it is necessary that' and 'it is possible that', by using classical propositional language and two modal operators: $\Box$ and $\Diamond$, for the terms 'necessary' and 'possible', respectively.

In fact, the symbols of modal operators $\Box$ and $\Diamond$ are not present in the first texts, but they were added later.

These days, by modal logic we understand a collection of other logics, that usually extend the classical logic by the introduction of new operators designed to formalize terms of modes; and a deontic logic emphasizes one of these cases, with special interest in legal subjects.

The deontic logic highlights three proper concepts to be investigated, obligatory (must), permitted and forbidden. In general, we choose 'obligatory' as a basic term and the others two are defined from that one.

We pick a basic deontic logic (*Standard deontic logic*), with the new operators, **O** for 'it is obligatory that', **P** for 'it is permitted that', and **F** for 'it is forbidden that'.

We take the basic system **KD** and present an algebraic model for the logic **KD**. In the following, we bring a short presentation of tableaux, a deduction system via refutation trees.

As an original contribution, we introduce a new tableaux system for **KD**.

There are other tableaux systems for **KD**, according to Kripke hierarchic tradition, as that ones which contemplate properties of the Kripke's structure $\langle W, R\rangle$, such that W is a non-empty set of possible worlds, and R is an accessibility relation for elements of W. The operators are interpreted in accordance to which properties are shared by the relation in that system.

We develop a distinct and simple system based in the algebraic model planned for **KD**.

2 Deontic basic logic

As we have mentioned, deontic logic is a particular case of modal logic, that prioritizes the deontic modality, with more interest for legal questions.

If a law determines an obligation, for example, that when driving all passengers must wear seat belts, the proposition is not true or false, but in a society that adopted that law, this conduct must be respected or the responsible ones can receive a penalty.

There is no just one deontic logic, but many. In this paper, we take one of them, the basic deontic logic, usually denoted by **KD**.

Any deontic logic investigates the terms obligatory, permitted, forbidden and all other concepts that can be defined from these ones.

Thus, we begin with a new unary operator, denoted by **O**, for 'it is obligatory that'.

From this new operator **O** we can, by definition, introduce the operators **P** for 'it is permitted that' and **F** for 'it is forbidden that':

$\mathbf{P}\varphi =_{df} \neg\mathbf{O}\neg\varphi$

$\mathbf{F}\varphi =_{df} \mathbf{O}\neg\varphi$.

An usual axiom system for the modal deontic system **KD** is the following one as [1], built over the classical propositional logic (**CPL**):

$\mathbf{KD}_1$:

(LPC)	φ, if φ is a tautology
(K)	$\mathbf{O}(\varphi \to \psi) \to (\mathbf{O}\varphi \to \mathbf{O}\psi)$
(D)	$\mathbf{O}\varphi \to \mathbf{P}\varphi$
(MP)	$\varphi \to \psi,\ \varphi \,/\, \psi$
(Nec)	$\vdash \varphi \,/\, \vdash \mathbf{O}\varphi$.

However, the system **KD** has several different presentations, as these that can be seen in the famous book of Chellas [2].

$\mathbf{KD}_2$:

(LPC)	φ, is φ is a tautology
(OD*)	$\neg(\mathbf{O}\varphi \wedge \mathbf{O}\neg\varphi)$
(MP)	$\varphi \to \psi,\ \varphi \,/\, \psi$
(ROK)	$\vdash (\varphi_1 \wedge \ldots \wedge \varphi_n) \to \psi \,/\, \vdash (\mathbf{O}\varphi_1 \wedge \ldots \wedge \mathbf{O}\varphi_n) \to \mathbf{O}\psi$, for $n \geq 0$.

$\mathbf{KD}_3$:

(LPC)	φ, if φ is a tautology
(OC)	$(\mathbf{O}\varphi \wedge \mathbf{O}\psi) \to \mathbf{O}(\varphi \wedge \psi)$
(ON)	$\mathbf{O}\top$
(OD)	$\neg\mathbf{O}\bot$
(MP)	$\varphi \to \psi,\ \varphi \,/\, \psi$
(ROM)	$\vdash \varphi \to \psi \,/\, \vdash \mathbf{O}\varphi \to \mathbf{O}\psi$.

Since we will take the system $\mathbf{KD}_3$, that has a presentation favourable to an algebraic characterization, we will show that these three axiomatic systems are equivalent.

Proposição 1 *The systems* $\boldsymbol{KD}_2$ *and* $\boldsymbol{KD}_3$ *are equivalent. In a first step, we will show that* ***OD**** *and* ***ROK*** *can be obtained in* $\boldsymbol{KD}_3$.

From ***OC*** *we have* $(\boldsymbol{O}\varphi \wedge \boldsymbol{O}\neg\varphi) \to \boldsymbol{O}(\varphi \wedge \neg\varphi) \Leftrightarrow (\boldsymbol{O}\varphi \wedge \boldsymbol{O}\neg\varphi) \to \boldsymbol{O}\bot$. *Now, considering* ***OD****, by contrapositive, we have* ***(OD*)****:* $\neg(\boldsymbol{O}\varphi \wedge \boldsymbol{O}\neg\varphi)$.

As in the ***CPL*** *it holds* $(\varphi \wedge \psi) \to \varphi$ *and* $(\varphi \wedge \psi) \to \psi$*, using the rule* ***ROM****, we get* $\boldsymbol{O}(\varphi \wedge \psi) \to (\boldsymbol{O}\varphi \wedge \boldsymbol{O}\psi)$. *This plus* ***OC*** *entail that* $\boldsymbol{O}(\varphi \wedge \psi) \leftrightarrow (\boldsymbol{O}\varphi \wedge \boldsymbol{O}\psi)$*, that can be seen as* $\boldsymbol{O}(\varphi_1 \wedge \ldots \wedge \varphi_n) \leftrightarrow (\boldsymbol{O}\varphi_1 \wedge \ldots \wedge \boldsymbol{O}\varphi_n)$.

Now, if we take $\vdash (\varphi_1 \wedge \ldots \wedge \varphi_n) \to \psi$*, using the rule* ***ROM*** *we will have* $\vdash \boldsymbol{O}(\varphi_1 \wedge \ldots \wedge \varphi_n) \to \boldsymbol{O}\psi$ *and, applying the equivalence above,* $\vdash (\boldsymbol{O}\varphi_1 \wedge \ldots \wedge \boldsymbol{O}\varphi_n) \to \boldsymbol{O}\psi$.

To complete, we must establish the other direction, that is, we must show that ***OC****,* ***ON****,* ***OD*** *and* ***ROM*** *are valid in* $\boldsymbol{KD}_2$.

With the rule ***ROK****, considering* $n = 0$*, we have* $\vdash \varphi \,/\, \vdash \boldsymbol{O}\varphi$. *As* $\vdash \top$*, then* $\vdash \boldsymbol{O}\top$.

Again with ***ROK*** *and now for* $n = 1$*, we verify the rule* ***ROM****.*

From the tautology $(\varphi \wedge \psi) \to (\varphi \wedge \psi)$*, with the rule* ***ROK*** *we get* ***OC****.*

From $\boldsymbol{OD^*}$ *and using the De Morgan law from* **CPL**: $\neg(\varphi \wedge \psi) \Leftrightarrow (\neg\varphi \vee \neg\psi)$, *we have* $\neg(\boldsymbol{O}\bot \wedge \boldsymbol{O}\neg\bot) \Leftrightarrow (\neg\boldsymbol{O}\bot \vee \neg\boldsymbol{O}\neg\bot)$, *or better,* $\neg(\boldsymbol{O}\bot \wedge \boldsymbol{O}\neg\bot) \Leftrightarrow (\neg\boldsymbol{O}\bot \vee \neg\boldsymbol{O}\top)$. *From* $\boldsymbol{ON}$ *and the disjunctive syllogism, then* $\neg\boldsymbol{O}\bot$.

Proposição 2 *The systems* $\boldsymbol{KD}_1$ *and* $\boldsymbol{KD}_2$ *are equivalent. The axioms of* $\boldsymbol{KD}_1$ *can be obtained from* $\boldsymbol{KD}_2$.

With **ROK**, *considering* $n = 0$, *we have the rule* **Nec**.

From $\boldsymbol{OD^*}$ *and the De Morgan law we generate* **D** *in the follow way:* $\neg(\boldsymbol{O}\varphi \wedge \boldsymbol{O}\neg\varphi) \Leftrightarrow (\neg\boldsymbol{O}\bot \vee \neg\boldsymbol{O}\neg\bot) \Leftrightarrow (\neg\boldsymbol{O}\bot \vee \boldsymbol{P}\varphi) \Leftrightarrow \boldsymbol{O}\varphi \rightarrow \boldsymbol{P}\varphi$.

Considering the tautology $((\varphi \rightarrow \psi) \wedge \varphi) \rightarrow \psi)$, *and using the rule* **ROK** *we get* $(\boldsymbol{O}(\varphi \rightarrow \psi) \wedge \boldsymbol{O}\varphi) \rightarrow \boldsymbol{O}\psi)$. *So, from* **CPL** *we have* **K**.

The axioms of $\boldsymbol{KD}_2$ *can be derived in* $\boldsymbol{KD}_1$.

As we proved above, the axioms $\boldsymbol{OD^*}$ *and* **D** *are equivalent.*

To obtain **ROK**, *we use* $\vdash (\varphi_1 \wedge \ldots \wedge \varphi_n) \rightarrow \psi$ *that with the rule* **Nec** *gives us* $\vdash \boldsymbol{O}((\varphi_1 \wedge \ldots \wedge \varphi_n) \rightarrow \psi)$ *and using the axiom* **K** *we have* $\vdash \boldsymbol{O}(\varphi_1 \wedge \ldots \wedge \varphi_n) \rightarrow \boldsymbol{O}\psi$. *With the equivalence,* $\boldsymbol{O}(\varphi_1 \wedge \ldots \wedge \varphi_n) \leftrightarrow (\boldsymbol{O}\varphi_1 \wedge \ldots \wedge \boldsymbol{O}\varphi_n)$, *we get* $\vdash (\boldsymbol{O}\varphi_1 \wedge \ldots \wedge \boldsymbol{O}\varphi_n) \rightarrow \boldsymbol{O}\psi$.

3 Algebraic model for KD

The algebraic model corresponds to a version of algebraic logic that should be considered. Since the logic **KD** is the **CPL** and something plus, then its correspondent algebra will be a Boolean algebra else some details associated with the deontic notions evolved.

As in the paper "Lógica deôntica básica e filtros" [3], of 2019, the algebraic model for **KD** is called **D**-algebra.

Definição 3 *A* **D**-*algebra is a structure* $\mathcal{D} = (D, 0, 1, \wedge, \vee, \sim, \xi)$, *such that* $(D, 0, 1, \wedge, \vee, \sim)$ *is a Boolean algebra and* $\xi : D \rightarrow D$ *is an unary operator for which:*

(i) $\xi 1 = 1$

(ii) $\xi 0 = 0$

(iii) $\xi(a \wedge b) = (\xi a \wedge \xi b)$.

The next semantic concepts will be important for the construction of tableaux for the deontic logic **KD** [3].

Definição 4 *A restrict valuation is a function* $\overline{v} : Var(\boldsymbol{KD}) \rightarrow \mathcal{A}$ *that maps each variable of* **KD** *in an element of a* **D**-*algebra* $\mathcal{A}$.

Definição 5 *A valuation is a function* $v : For(\boldsymbol{KD}) \rightarrow \mathcal{A}$, *that extends naturally and exclusively* $\overline{v}$ *in the following way:*

(i) $v(p) = \overline{v}(p)$

(ii) $v(\neg\varphi) = \sim v(\varphi)$

(iii) $v(\varphi \wedge \psi) = v(\varphi) \wedge v(\psi)$

(iv) $v(\varphi \vee \psi) = v(\varphi) \vee v(\psi)$

(v) $v(\boldsymbol{O}\varphi) = \xi v(\varphi))$.

The symbols $\neg$, $\wedge$, $\vee$ and **O** in the left represent logical operators, while the symbols $\sim$, $\wedge$, $\vee$, ξ in the right represent algebraic operators of $\mathcal{A}$.

Definição 6 *A valuation* $v : For(\textbf{\textit{KD}}) \to \mathcal{A}$ *is a model for a set* $\Gamma \subseteq For(\textbf{\textit{KD}})$ *if* $v(\varphi) = 1$*, for each formula* $\varphi \in \Gamma$.

In particular, a valuation $v : For(\textbf{KD}) \to \mathcal{A}$ is a model for a formula $\varphi \in For(\textbf{KD})$ when $v(\varphi) = 1$.

Definição 7 *A formula* $\varphi \in For(\textbf{\textit{KD}})$ *is valid in a* ***D****-algebra* $\mathcal{A}$ *if every valuation* $v : For(\textbf{\textit{KD}}) \to \mathcal{A}$ *is model for* φ.

Definição 8 *A formula is* ***D****-valid when it is valid in any* ***D****-algebra, what is denoted by* $\vDash \varphi$.

More details about definitions, theorems and proofs relative to the algebra of **KD** can be seen in [3].

4 Tableaux

The tableaux method is a very simple, elegant and fast way to produce proofs or deductions.

Let's see only some notions for the tableaux of classical propositional logic **CPL**, as we can see in [4], [5], and mainly [6].

The tableaux system for **KD** will be an extension of the tableaux for **CPL**.

Since the tableaux configure a refutation procedure, the following theorem puts some light on it.

Teorema 9 *If* $\Gamma \cup \{\alpha\}$ *is a set of formulas of* ***CPL****, then* $\Gamma \vDash \alpha$ *iff* $\Gamma \cup \{\neg\alpha\}$ *is non-satisfiable. A proof can be met in [7].*

Since the tableaux for **CPL** are very usual and well known, we will just present the expansion rules and give two examples.

We will use a notation with marked formulas or signed formulas, with T for true and F for false.

- **Expansion Rules**:

(**RC1**): $\dfrac{\mathbf{T}\ \neg\varphi}{\mathbf{F}\ \varphi}$

(**RC2**): $\dfrac{\mathbf{F}\ \neg\varphi}{\mathbf{T}\ \varphi}$

(**RC3**): $\dfrac{\mathbf{T}\ (\varphi \wedge \psi)}{\begin{array}{c}\mathbf{T}\ \varphi \\ \mathbf{T}\psi\end{array}}$

(**RC4**): $\dfrac{\mathbf{F}\ (\varphi \wedge \psi)}{\mathbf{F}\ \varphi \quad \curlywedge \quad \mathbf{F}\ \psi}$

(**RC5**): $$\frac{\mathbf{T}\ (\varphi \vee \psi)}{\mathbf{T}\ \varphi \quad \wedge \quad \mathbf{T}\ \psi}$$

(**RC6**): $$\frac{\mathbf{F}\ (\varphi \vee \psi)}{\begin{array}{c}\mathbf{F}\ \varphi \\ \mathbf{F}\ \psi\end{array}}$$

(**RC7**): $$\frac{\mathbf{T}\ (\varphi \rightarrow \psi)}{\mathbf{F}\ \varphi \quad \wedge \quad \mathbf{T}\ \psi}$$

(**RC8**): $$\frac{\mathbf{F}\ (\varphi \rightarrow \psi)}{\begin{array}{c}\mathbf{T}\ \varphi \\ \mathbf{F}\ \psi\end{array}}$$

(**RC9**): $$\frac{\mathbf{T}\ (\varphi \leftrightarrow \psi)}{\begin{array}{c}\mathbf{T}\ \varphi \\ \mathbf{T}\ \psi\end{array} \quad \wedge \quad \begin{array}{c}\mathbf{F}\ \varphi \\ \mathbf{F}\ \psi\end{array}}$$

(**RC10**): $$\frac{\mathbf{F}\ (\varphi \leftrightarrow \psi)}{\begin{array}{c}\mathbf{T}\ \varphi \\ \mathbf{F}\ \psi\end{array} \quad \wedge \quad \begin{array}{c}\mathbf{F}\ \varphi \\ \mathbf{T}\ \psi\end{array}}$$

We denote a path in a tableau by η.

Some examples, the first for a valid formula and the second, with an open path, for a non-valid formula:

(1) $((\varphi \rightarrow (\psi \wedge \sigma)) \wedge \neg\sigma) \rightarrow \neg\varphi$

$$\begin{array}{c}
\mathbf{F} \quad ((\varphi \rightarrow (\psi \wedge \sigma)) \wedge \neg\sigma) \rightarrow \neg\varphi \\
\mathbf{V} \quad (\varphi \rightarrow (\psi \wedge \sigma)) \wedge \neg\sigma \\
\mathbf{F} \quad \neg\varphi \\
\mathbf{V} \quad \varphi \\
\mathbf{V} \quad \varphi \rightarrow (\psi \wedge \sigma) \\
\mathbf{V} \quad \neg\sigma \\
\mathbf{F} \quad \sigma \\
\swarrow \qquad \searrow \\
\begin{array}{c} \mathbf{F} \quad \varphi \\ \times \\ \\ \\ \end{array} \qquad\qquad
\begin{array}{c} \mathbf{V} \quad \psi \wedge \sigma \\ \mathbf{V} \quad \psi \\ \mathbf{V} \quad \sigma \\ \times \end{array}
\end{array}$$

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(2) $((\varphi \to \psi) \to \psi) \to \psi$

$$
\begin{array}{ccc}
 & \mathbf{F} \quad ((\varphi \to \psi) \to \psi) \to \psi & \\
 & \mathbf{V} \quad (\varphi \to \psi) \to \psi & \\
 & \mathbf{F} \quad \psi & \\
 \swarrow & & \searrow \\
\mathbf{F} \quad \varphi \to \psi & & \mathbf{V} \quad \psi \\
\mathbf{V} \quad \varphi & & \vdots \\
\mathbf{F} \quad \psi & & \times
\end{array}
$$

A valuation v such that $v(\varphi) = V$ and $v(\psi) = F$ forces the initial formula to be false.

5 A tableaux system for KD

In this section, we introduce the original element of this paper, a new tableaux system for the logic **KD**.

The tableaux system for **KD** will be built from elements and rules for **CPL**, plus some new specific rules for the new operator '**O**', considering its correlated axioms.

We denote the new tableaux system for **KD** by $\mathcal{T}_{L(D)}$.

Definição 10 *A path of a tableau in $\mathcal{T}_{L(D)}$ is closed if the marked formulas occur in the path:*
(i) ***T*** *α and* ***F*** *α, for any formula α.*
(ii) ***T O****α and* ***F O****α, for some formula* ***O****α.*

Definição 11 *A tableau in $\mathcal{T}_{L(D)}$ is closed if all of its paths are closed.*

All the tableaux rules for **CPL** are valid.

Now, we introduce specific rules for the operator **O**, considering the axiomatic systems of **KD** and also the D-algebras.

Since the usual modal axiom **(T)**: $\mathbf{O}\varphi \to \varphi$ is not valid, then we cannot reduce modal formulas of type $\mathbf{O}\varphi$ to the sub-formula φ. For if it is obligatory that φ it is not necessary that φ.

(R_1)

$$
\begin{array}{ll}
\mathbf{T} & \mathbf{O}(\neg\varphi) \\
 & \quad | \\
\mathbf{F} & \quad \mathbf{O}\,\varphi
\end{array}
$$

If a law is obligatory, then its negation cannot be obligatory. However, if a law is not obligatory, we cannot say anything about its negation.

The next two rules come from properties of D-algebras: $\xi(a \wedge b) = (\xi a \wedge \xi b)$.

(R_2)

$$
\begin{array}{ll}
\mathbf{T} & \mathbf{O}(\varphi \wedge \psi) \\
 & \quad | \\
\mathbf{T} & \quad \mathbf{O}\,\varphi \\
 & \quad | \\
\mathbf{T} & \quad \mathbf{O}\,\psi
\end{array}
$$

(R_3)

$$\begin{array}{ccc} & \mathbf{F} \quad \mathbf{O}(\varphi \wedge \psi) & \\ \swarrow & & \searrow \\ \mathbf{F} \ \ \mathbf{O}\,\varphi & & \mathbf{F} \ \ \mathbf{O}\,\psi \end{array}$$

(R_4)

$$\begin{array}{cc} \mathbf{F} & \mathbf{O}(\varphi) \\ & | \\ \mathbf{F} & \varphi \end{array}$$

The rule (R_4) elapses from contrapositive of **NEC**.

(R_5)

If $\Vdash \varphi \rightarrow \psi$ and $\mathbf{T} \quad \mathbf{O}(\varphi)$, then we include in the path $\mathbf{T} \quad \mathbf{O}(\psi)$.

The rule (R_5) is only applied when the conditional $\varphi \rightarrow \psi$ is indeed valid and not just in case it occurs in the tableau marked with **T**. Neither every conditional must be valid.

Some examples in the tableaux system for **KD**.

(1) $\mathbf{O}\neg\varphi \rightarrow (\varphi \rightarrow \mathbf{O}\psi)$

$$\begin{array}{c} \mathbf{F} \quad \mathbf{O}\neg\varphi \rightarrow (\varphi \rightarrow \mathbf{O}\psi) \\ \mathbf{T} \quad \mathbf{O}\neg\varphi \\ \mathbf{F} \quad \varphi \rightarrow \mathbf{O}\psi \\ \mathbf{F} \quad \mathbf{O}\varphi \\ \mathbf{T} \quad \varphi \\ \mathbf{F} \quad \mathbf{O}\psi \\ \mathbf{F} \quad \varphi \\ \times \end{array}$$

The above formula can be understood as the theorem of derived obligation. If it is forbidden that φ and you commit φ, then you are subject to a consequence.

A view about the tableau.

We apply (R_1) from the second line to the fourth line. Then we apply (R_4) from the fourth to the seventh line, and get the contradiction $\mathbf{T}\ \varphi$ e $\mathbf{F}\ \varphi$. Hence, the tableau is closed.

As usual, the formula $\neg(\mathbf{O}\neg\varphi \rightarrow (\varphi \rightarrow \mathbf{O}\psi))$ cannot be valid and, then, its negation $\mathbf{O}\neg\varphi \rightarrow (\varphi \rightarrow \mathbf{O}\psi)$ is true.

(2) $(\neg\mathbf{O}\neg\varphi \rightarrow \mathbf{O}\psi) \rightarrow (\mathbf{O}\varphi \rightarrow \mathbf{O}\psi)$

$$\begin{array}{c} \mathbf{F} \quad (\neg\mathbf{O}\neg\varphi \rightarrow \mathbf{O}\psi) \rightarrow (\mathbf{O}\varphi \rightarrow \mathbf{O}\psi) \\ \mathbf{T} \quad \neg\mathbf{O}\neg\varphi \rightarrow \mathbf{O}\psi \\ \mathbf{F} \quad \mathbf{O}\varphi \rightarrow \mathbf{O}\psi \\ \mathbf{T} \quad \mathbf{O}\varphi \\ \mathbf{F} \quad \mathbf{O}\psi \end{array}$$

$$
\begin{array}{ccc}
 & \diagup & \diagdown \\
\mathbf{F} \quad \neg\mathbf{O}\neg\varphi & & \mathbf{T} \quad \mathbf{O}\psi \\
\mathbf{T} \quad \mathbf{O}\neg\varphi & & \bot \\
\mathbf{F} \quad \mathbf{O}\varphi & & \vdots \\
\times & & \times
\end{array}
$$

From the seventh line on left we apply (R_1) and close the path with the contradiction **T** $\mathbf{O}\varphi$ and **F** $\mathbf{O}\varphi$. In the right path we get the contradiction **T** $\mathbf{O}\psi$ and **F** $\mathbf{O}\psi$.

(3) $\mathbf{O}(\varphi \wedge \psi) \rightarrow \neg\mathbf{O}\neg\varphi$

$$
\begin{array}{c}
\mathbf{F} \quad \mathbf{O}(\varphi \wedge \psi) \rightarrow \neg\mathbf{O}\neg\varphi \\
\mathbf{T} \quad \mathbf{O}(\varphi \wedge \psi) \\
\mathbf{F} \quad \neg\mathbf{O}\neg\varphi \\
\mathbf{T} \quad \mathbf{O}\varphi \\
\mathbf{T} \quad \mathbf{O}\psi \\
\mathbf{T} \quad \mathbf{O}\neg\varphi \\
\mathbf{F} \quad \mathbf{O}\varphi \\
\times
\end{array}
$$

We use (R_2) from the second to the fourth and fifth lines. Then we apply (R_1) from the sixth to the seventh line. We obtain the contradiction **T** $\mathbf{O}\varphi$ and **F** $\mathbf{O}\varphi$ in a single path.

5.1 From deductive system to tableaux system

We must show the equivalence between some axiomatic system for **KD** and the tableaux system $\mathcal{T}_{L(D)}$, that we have introduced.

First, we will show that for each axiomatic deduction in $\mathbf{KD}_3$ there is a correspondent deduction in the tableaux system $\mathcal{T}_{L(D)}$.

Teorema 12 *If* $\Gamma \vdash \gamma \Rightarrow \Gamma \Vdash \gamma$. *By induction in the length of deduction* $\Gamma \vdash \gamma$.
If $n = 1$, *then or* $\gamma \in \Gamma$, *or* γ *is an axiom of* ***KD***.
If $\gamma \in \Gamma$, *then* $\Gamma \Vdash \gamma$, *for* γ *occur with distinct values in the tableau and the tableau is closed.*

If γ *is an axiom, then we will show that* $\Vdash \gamma$ *and, this way,* $\Gamma \Vdash \gamma$.

- for (ON), $\boldsymbol{O}\top$*:*

$$
\begin{array}{c}
\boldsymbol{F} \ \ \boldsymbol{O}\top \\
\boldsymbol{F} \ \ \top \\
\boldsymbol{F} \ \ \varphi \vee \neg\varphi \\
\boldsymbol{F} \ \ \varphi \\
\boldsymbol{F} \ \ \neg\varphi \\
\boldsymbol{T} \ \ \varphi \\
\times
\end{array}
$$

We used (R_4) *from the first to the second line of the tableau.*

- For (OD), $\neg\boldsymbol{O}\bot$*:*

$$
\begin{array}{c}
\boldsymbol{F}\ \neg\boldsymbol{O}\bot \\
\boldsymbol{T}\ \boldsymbol{O}\bot \\
\boldsymbol{T}\ \boldsymbol{O}\neg\top \\
\boldsymbol{F}\ \boldsymbol{O}\top \\
\times
\end{array}
$$

We used (R_1) from the third to the fourth line of the tableau. Then the tableau closed in according (ON).

- For (OC), $(\boldsymbol{O}\varphi \wedge \boldsymbol{O}\psi) \rightarrow \boldsymbol{O}(\varphi \wedge \psi)$:

$$
\begin{array}{ccc}
 & \boldsymbol{F}\ (\boldsymbol{O}\varphi \wedge \boldsymbol{O}\psi) \rightarrow \boldsymbol{O}(\varphi \wedge \psi) & \\
 & \boldsymbol{T}\ \boldsymbol{O}\varphi \wedge \boldsymbol{O}\psi & \\
 & \boldsymbol{F}\ \boldsymbol{O}(\varphi \wedge \psi) & \\
 & \boldsymbol{T}\ \boldsymbol{O}\varphi & \\
 & \boldsymbol{T}\ \boldsymbol{O}\psi & \\
 & \swarrow \quad \searrow & \\
\boldsymbol{F}\ \boldsymbol{O}\varphi & & \boldsymbol{F}\ \boldsymbol{O}\psi \\
\times & & \times
\end{array}
$$

We used (R_3) from the third to the sixth line of the tableau.

- for (ROM), $\vdash \varphi \rightarrow \psi \,/ \vdash \boldsymbol{O}\varphi \rightarrow \boldsymbol{O}\psi$:

$$
\begin{array}{ccc}
 & \boldsymbol{T}\ \Vdash \varphi \rightarrow \psi & \\
 & \boldsymbol{F}\ \boldsymbol{O}\varphi \rightarrow \boldsymbol{O}\psi & \\
 & \boldsymbol{T}\ \boldsymbol{O}\varphi & \\
 & \boldsymbol{F}\ \boldsymbol{O}\psi & \\
 & \boldsymbol{F}\ \psi & \\
 & \swarrow \quad \searrow & \\
\boldsymbol{F}\ \varphi & & \boldsymbol{T}\ \psi \\
| & & \bot \\
\boldsymbol{T}\ \boldsymbol{O}\psi & & \times
\end{array}
$$

$\times$

We have used (R_1) from the fourth to the fifth line and (R_5) from lines 1 and 3 to the line 7.

With the above tableaux we have proved that if $\Gamma \vdash \gamma$, then $\Gamma \Vdash \gamma$, or yet, that the tableau of $\Gamma \cup \{\neg\gamma\}$ is closed.

Thus, everything that we can deduce in some axiomatic system for **KD** can be deduced in the tableaux system $\mathcal{T}_{L(D)}$.

But would be the tableaux system more powerful than the axiomatic systems? No, they must have exactly the same deductive power.

For that we will show the reverse of the last theorem, but we will use the algebraic deduction for **KD**.

5.2 From tableaux system for algebraic consequence

For the conclusion of he equivalence between our tableaux system and the logic **KD**, we must show that: $\Gamma \Vdash \varphi \Rightarrow \Gamma \vDash \varphi$.

However, to get the proof we need to introduce some essential results and definitions.

The algebraic consequence holds for any D-algebra, that is, a Boolean algebra plus the properties of the deontic operator. Now, we consider the Boolean algebra of two elements $\mathbf{2} = \{F, T\}$.

Teorema 13 *Every rule of the system $\mathcal{T}_{L(D)}$ leads true premises in true consequences.*

This result is well known for the Boolean rules, as we can see in [6].

*We will show the validity of five new rules in $\mathcal{T}_{L(D)}$. Since we have the strong adequacy of **KD** and the algebraic models of D-algebras, then we present only deductions here.*

- For the rule (R_1), we show that $\Gamma \vdash \boldsymbol{O}(\neg\varphi) \Rightarrow \Gamma \vdash \neg\boldsymbol{O}\varphi$:

1. $\Gamma \vdash \boldsymbol{O}(\neg\varphi)$	*Hypothesis*
2. $\Gamma \vdash \boldsymbol{O}(\varphi \to \bot)$	*Substitution*
3. $\Gamma \vdash \boldsymbol{O}\varphi \to \boldsymbol{O}\bot$	*ROM in (2)*
4. $\Gamma \vdash \neg\boldsymbol{O}\bot$	*OD*
5. $\Gamma \vdash \neg\boldsymbol{O}\varphi$	*MT in (3) and (4)*

- For the rule (R_2), we show that $\Gamma \vdash \boldsymbol{O}(\varphi \wedge \psi) \Rightarrow \Gamma \vdash \boldsymbol{O}\varphi \wedge \boldsymbol{O}\psi$:

1. $\Gamma \vdash \boldsymbol{O}(\varphi \wedge \psi)$	*Hypothesis*
2. $\Gamma \vdash (\varphi \wedge \psi) \to \varphi$	*CPL*
3. $\Gamma \vdash \boldsymbol{O}(\varphi \wedge \psi) \to \boldsymbol{O}\varphi$	*ROM in (2)*
4. $\Gamma \vdash (\varphi \wedge \psi) \to \psi$	*CPL*
5. $\Gamma \vdash \boldsymbol{O}(\varphi \wedge \psi) \to \boldsymbol{O}\psi$	*ROM in (4)*
6. $\Gamma \vdash \boldsymbol{O}\varphi$	*MP in (1) and (3)*
7. $\Gamma \vdash \boldsymbol{O}\psi$	*MP in (1) and (5)*
8. $\Gamma \vdash \boldsymbol{O}\varphi \wedge \boldsymbol{O}\psi$	*Conjunction in (6) and (7)*

- For the rule (R_3), we will show that $\Gamma \vdash \neg\boldsymbol{O}(\varphi \wedge \psi) \Rightarrow \Gamma \vdash \neg\boldsymbol{O}\varphi \vee \neg\boldsymbol{O}\psi$:

1. $\Gamma \vdash \neg\boldsymbol{O}(\varphi \wedge \psi)$	*Hypothesis*
2. $\Gamma \vdash (\boldsymbol{O}\varphi \wedge \boldsymbol{O}\psi) \to \boldsymbol{O}(\varphi \wedge \psi)$	*OC*
3. $\Gamma \vdash \neg(\boldsymbol{O}\varphi \wedge \boldsymbol{O}\psi)$	*MT in (1) and (2)*
4. $\Gamma \vdash \neg\boldsymbol{O}\varphi \vee \neg\boldsymbol{O}\psi$	*De Morgan in (3)*

- For the rule (R_4), we must show that $\Gamma \vdash \neg\boldsymbol{O}\varphi \Rightarrow \Gamma \vdash \neg\varphi$:

1. $\Gamma \vdash \neg\boldsymbol{O}\varphi$	*Hypothesis*
2. $\Gamma \vdash \neg\neg\varphi$	*p.p.*
3. $\Gamma \vdash \varphi$	*DN in (2)*
4. $\Gamma \vdash \boldsymbol{O}\varphi$	*Nec in (3)*
5. $\Gamma \vdash \boldsymbol{O}\varphi \wedge \neg\boldsymbol{O}\varphi$	*Conjunction in (1) and (4)*
6. $\Gamma \vdash \bot$	*Absurd Reduction from (2) to (5)*

- For the rule (R_5), we must show that $\Gamma \vdash \boldsymbol{O}\varphi$ e $\vdash \varphi \to \psi$, then $\Gamma \vdash \boldsymbol{O}\psi$:

1. $\Gamma \vdash \varphi \to \psi$	*Hypothesis*
2. $\Gamma \vdash \boldsymbol{O}\varphi$	*Hypothesis*

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3. $\Gamma \vdash \boldsymbol{O}\varphi \to \boldsymbol{O}\psi$ *ROM in (1)*
5. $\Gamma \vdash \boldsymbol{O}\psi$ *MP in (2) and (3).*

Thus, we verified that the rules of the system $\mathcal{T}_{L(D)}$ preserve deduction and validity.

In the following we will show that $\Gamma \Vdash \gamma \Rightarrow \Gamma \vDash \gamma$ and, considering the algebraic adequacy of **KD**, the axiomatic version of **KD** and $\mathcal{T}_{L(D)}$, its tableaux version, are deductively equivalent.

Definição 14 *A set of marked formulas Θ is down-saturated if it satisfies the following conditions:*
(a) *no marked formula occurs in Θ with distinct values;*
(b) *if some marked formula α of type A occurs in Θ, then $\alpha_1 \in \Theta$ and $\alpha_2 \in \Theta$, such that α_1 and α_2 are marked formulas and they are immediate of α, in conformity with its respective rule;*
(c) *if some marked formula β of type B occurs in Θ, then $\beta_1 \in \Theta$ or $\beta_2 \in \Theta$, for $\beta 1$ and $\beta 2$ are marked formulas and are immediate of β, according to its respective rule.*

Lema 15 *Every path saturated and open in a tableau is a down-saturated set. As the path is open, then no formula appears in the path with distinct valuations, what satisfies the condition (a) of Definition 14.*
Besides, as the path is saturated, it follows that all the possible rules were applied and the tableau can be expanded no more.
So, if there is a formula of type A in the path, then α_1 and α_2 are also in the path, what meets the condition (b).
By the same reason, if there is a formula of type B in the path, then or β_1, or β_2 is in the path, what fulfils the condition (c).

Now, we extend the notion of valuation for marked formulas.

Definição 16 *If v is a valuation and $k \in \{F, T\}$, then the marked formula $k\ \varphi$ is distinguished according to the valuation v, what is denoted by $k\ \varphi \in D$, if $v(\varphi) = k$.*

Thus, $k\ \varphi \in D \Leftrightarrow v(\varphi) = k$.

Definição 17 *A valuation v satisfies a set Θ of marked formulas if for every marked formula $k\ \psi$ that occurs in Θ, we have that $k\ \psi \in D$.*

Definição 18 *A set of marked formulas Θ is satisfiable if there is a valuation v such that $v(\Theta) \subseteq \{V\}$, that is, for every $\psi \in \Theta$, $V\ \psi \in D$.*

Lema 19 *If Θ is a satisfiable set of marked formulas, then:*
(i) *if a formula α of type A is in Θ, then $\Theta \cup \{\alpha_1, \alpha_2\}$ is satisfiable;*
(ii) *if a formula β of type B is in Θ, then $\Theta \cup \{\beta_1\}$ is satisfiable or $\Theta \cup \{\beta_2\}$ is satisfiable. This result is a immediate consequence of Theorem 13, for the new rules for the deontic operator, when applied, preserve the validity.*
(i) Rules of type A: (R_1), (R_2), (R_3) and (R_4).
(ii) Rules of type B: (R_3).

From these considerations and the above lemma we can enunciate and to prove the next theorem.

Teorema 20 $\Gamma \Vdash \varphi \Rightarrow \Gamma \vDash \varphi$. *The proof is by contrapositive.*

If $\Gamma \nvDash \varphi$, then there exists a valuation v, such that $v(\Gamma) \subseteq \{T\}$ and $v(\varphi) = F$.

Let Θ_0 be the set of marked formulas that appear in the beginning tableau of Γ, such that $v(\Theta_0) \subseteq D$. We show that in each step of tableau expansion, there will always be a path Θ_0 such that $v(\Theta_0) \subseteq D$.

Suppose that $v(\Theta_{i-1}) \subseteq D$. If the path Θ_{i-1} is expanded by a formula of type A, by previous lemma (i), it follows that$v(\Theta_i) \subseteq D$.

In the case the path Θ_{i-1} is expanded by a formula of type B, from the lemma (ii), we have that $v(\Theta_i) \subseteq D$.

Thus, in all cases, we have a path Θ_i such that $v(\Theta_i) \subseteq D$. This way, there will always be a satisfiable path in Θ, which is a down-saturated set.

Hence, $\Gamma \nVdash \varphi$

Finally, with these results we have introduced a very simple tableaux system that is deductively equivalent to any axiomatic system for the logic **KD**.

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