

**Revista Eletrônica
Paulista de Matemática**

ISSN 2316-9664

v. 22, n. 2, set. 2022

Edição Brazilian Symposium on
Fractional Calculus

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And history repeats itself... Fractional tree

Abstract

This article aims to present some important points within the evolution of fractional calculus, which today in the field of scientific research has become a field of great relevance with its important contributions to the development of science. It is important to note that this article is a transcription of the lecture given by Professor E. Capelas de Oliveira as the closing lecture of the II Brazilian Congress of Fractional Calculus, which gives a brief explanation of fractional calculus and some of their contributions throughout his career at Unicamp, addressed to its former master and doctoral students. Even more, as the lecture was given in Portuguese, a lot is lost in the transcription to another language.

Keywords: Fractional calculus. Chronology. Fractional tree.



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1 Introduction

I was invited by the organizing committee of the II Brazilian Symposium on Fractional Calculus to present the closing lecture of the symposium, an honour that I accepted with great pleasure.

Before starting, I remember that the lecture was presented in Portuguese. Thus, part of its original spirit was lost due to the translation work and the changes necessary to transform the lecture into this article, which aims to document the history of a seed that sprouted, branched and generated many trees in several states of Brazil.

Although it is not a simple task to present the closing lecture of a symposium, this work was both challenging and motivating, and I enjoyed the work of choosing the best approach and preparing the lecture. In particular, because it was probably my last lecture, as I am retiring. After listening to some suggestions, I decided to take that unique opportunity to express my gratitude to the students, now teachers, who walked the path of fractional calculus along with me.

Once the approach had been chosen, I divided the lecture in three parts. The first part covers the period from the beginnings of fractional calculus up to the date of the symposium, with emphasis on year 2005, when the study of fractional calculus began at Campinas. Of course, in such a brief lecture, many relevant facts and people had to be put aside. Thus, the resulting timeline, even though it is correct, is not exhaustive nor unique and involves choices which reflect the evolution of mine and our student's views on the subject since we began to learn it until we could take part in its development.

The second part returns to year 2005. Starting from that year, we present the students who graduated under my guidance or co-orientation, as well as their corresponding fruits, students formed by my former students, now teachers. The third part summarizes what was the II Brazilian Symposium on Fractional Calculus, carried out virtually, highlighting its collaborators, invited speakers and participants.

Finally, I concluded my speech with a simple scheme which I called *fractional tree*, a kind of pseudosystem in which each former student becomes a researcher and acquires a life of his/her own, a reference to Plato's regular polygons. I start with the equilateral triangle, beginning in 2005, and go through the square —first doctoral thesis—, the hexagon —students who completed their master's degree—, arriving at the dodecagon —students who completed their doctorate— and aiming at the icosagon. . .

2 A short chronology and location

The following brief and simple chronology attempts to cover more than 300 years of history of fractional calculus in no more than thirty entries. I tried to describe the origin and evolution of the group formed at Unicamp, Campinas, to study fractional calculus. I begin with the (famous) correspondence between Leibniz and l'Hôpital, presenting what I consider the milestones in the path that led to the appearance of our group. It was not my intention to present a detailed chronology nor to cover topics which, for one reason or another, lay outside the scope of our research.

- 1695 – Leibniz $\rightleftharpoons$ l'Hôpital (30 september!) There are several books and articles in which we find the date of September 30th as the beginning of non integer order calculus (Miller & Ross). Although it is true that there was an exchange of correspondence between Leibniz and l'Hôpital, there is no guarantee that a letter dated September 30th contained a discussion about the meaning of a derivative of non integer order, in this case, a derivative of order $1/2$, a fraction. This myth lasted for a long time, as we will see a little later. In my opinion, the

most important fact to mention is that non integer order calculus, fractional calculus, is as old as integer order differential and integral calculus, as proposed by Newton and Leibniz, whose rigorous formulation was accomplished only with the works of Cauchy and Weierstrass.

- 1847 – Riemann-Liouville → Fractional integral. Although Liouville has been the greatest exponent of fractional calculus of 19th century, a subject to which he gave many contributions, the expression for a fractional integral became known as the Riemann-Liouville integral. In this first expression, Riemann introduced a characteristic function in order to take the lower end into account:

$${}_c J_x^\alpha [f(t)](x) = \frac{1}{\Gamma(\alpha)} \int_c^x (x-t)^{\alpha-1} f(t) dt.$$

From this integral we get the so-called Riemann-Liouville derivative,

$$D^\alpha f(x) = D^n [{}_c J_x^\alpha f(x)],$$

where $n \in \mathbb{N}$ is such that $n-1 < \alpha \leq n$. This definition, in words, turns out to be an integer order derivative of an integral of non integer order. Note that this formula contains the constant c , the lower limit of the integral. If this limit is $-\infty$, we have a Liouville derivative; in the case in which the constant is zero, we have a Riemann derivative.

- 1969 – Caputo → Caputo derivative. Only after 275 years did a profound change take place, with the introduction of what is now known as the Caputo derivative, proposed by himself when studying viscoelasticity problems, namely,

$${}^C D^\alpha f(x) \equiv D_*^\alpha f(x) = J_x^\alpha [D^n f(x)],$$

which, in words, calculates the non integer order derivative from the non integer order integral of an integer order derivative. We emphasize that Caputo's formulation is more restrictive, as it requires that the function admits a derivative of integer order.

It should be noted that this derivative was also introduced by Dzerbashyan, possibly a little time before Caputo. There is much controversy around this fact because at that time communication did not have the help of internet. Moreover, Dzerbashyan published his article in Russian while Caputo published his in Italian, then both were translated into English. An important property that deserves mention is that the Caputo derivative of a constant is always zero, which is not the case with the Riemann-Liouville derivative!

- 1974 – New Haven → First specialized conference. The *International Conference on Fractional Calculus and its Applications* was entirely dedicated to fractional calculus and took place at the University of New Haven, USA. The proceedings were published in 1975: B. Ross. *Proceedings of the International Conference on Fractional Calculus and its Applications*. DOI 10.1007/BFb0067095. The first monograph dedicated to fractional calculus was published in the same year: K. Oldham and J. Spanier, *The Fractional Calculus. Theory and Applications of Differentiation and Integration to Arbitrary Order*, Academic Press, INC, San Diego, 1974. Moreover, in this year a criterion was proposed for classifying fractional derivatives
- 1987 – S. Samko, O. Marichev and A. Kilbas published the classical book *Fractional Integrals and Derivatives and some of their Applications*, in Russian. Definitely one of the most important books on the subject. This book was translated into English in 1993.

- 1993 – K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, Inc., New York. An important book in which the authors present a large timeline and discuss several fractional differential equations.
- 1994 – S. Dugowson → Les différentielles métaphysiques: histoire et philosophie de la généralisation de l'ordre de dérivation (Thèse de doctorat en Mathématiques). Almost 300 years after the beginnings of fractional calculus, the author brings to light several details of the Leibniz-L'Hôpital correspondence, including the original question at the origin of non integer order calculus. It was launched the most important journal on fractional calculus, *Fractional Calculus and Applied Analysis*, edited by Virginia Kiryakova.
- 1996 – Kolwankar → Local fractional calculus. In this year, an article was published in which the so-called local fractional calculus was introduced. However, it has been shown that the derivative proposed by the author cannot be considered a classical fractional derivative.
- 2000 – FRACALMO → FRActional CALculus MOdelling. This was the first site devoted to the discussion of some problems of fractional calculus. Today, due to the ubiquity of Internet, the site is not kept up-to-date.
- 2005 – Vinícius Alegreti Meza, Rubens Figueiredo Camargo and Edmundo Capelas de Oliveira. This year, the study of fractional calculus began at Campinas. See below the *equilateral triangle*.
- 2010 – J. A. Tenreiro Machado, V. Kiryakova, F. Mainardi, *A Poster About the Recent History of Fractional Calculus*, *Fract. Calc. & Appl. Anal.*, **13**, 329-334. In this important article, an A3-Poster can be downloaded; it contains, as its name suggests, important researchers and books on fractional calculus. I printed a poster and donated it to the library of Imecc (Instituto de Matemática, Estatística e Computação Científica), in Campinas.
- 2010 – J. A. Tenreiro Machado, V. Kiryakova, F. Mainardi, *A Poster About the Old History of Fractional Calculus*, *Fract. Calc. & Appl. Anal.*, **13**, 447-454. Another A3-Poster that can be downloaded; it presents early researchers and books on fractional calculus. I also printed a copy and donated it to the library of Imecc (Instituto de Matemática, Estatística e Computação Científica), Campinas.
- 2013 – Rubens → I BSFC. Rubens de Figueiredo Camargo organizes the First Brazilian Symposium on Fractional Calculus at Universidade Estadual Paulista, Bauru. In that symposium important names of fractional calculus were present, among them: Francesco Mainardi, University of Bologna, Italy; José António Tenreiro Machado, University of Porto, Portugal; Ervin Kaminski Lenzi, Universidade Federal de Ponta Grossa, Brazil; Matheus Jatkoske Lazo, Universidade Federal do Rio Grande, Rio Grande do Sul, Brazil and Jayme Vaz Júnior, Universidade Estadual de Campinas, São Paulo, Brazil. During this meeting, the First Brazilian Minisymposium on Non Integer Order Calculus and Applications was held, in honour of Prof. Dr. Edmundo Capelas de Oliveira.
- 2014 – F. S. Costa, E. C. Grigoletto, J. Vaz Jr, E. Capelas de Oliveira, *Slowing-down of neutrons: a fractional model*, *Commun. Appl. Math.*, **6** (2), 15-30. This paper was published as a tribute to prof. Francesco Mainardi in the occasion of his retirement and also for celebrating his 70th birthday.

- 2015 – Ortigueira and Tenreiro Machado → Criterion. M. D. Ortigueira and J. A. Tenreiro Machado, *What is a fractional derivative?*, J. Comput. Phys., **293**, 4-13. Special issue on Fractional Partial Differential Equations. This paper contains a criterion for the definition of a fractional derivative more restrictive than the criterion proposed by Ross forty years earlier. The authors excluded the requirement of analyticity appearing in Ross's criterion and replaced it with Leibniz rule.
- 2015 – Rubens + Edmundo → Fractional calculus. The first book on fractional calculus written in Portuguese was published in Brazil: R. Figueiredo Camargo and E. Capelas de Oliveira, *Cálculo Fracionário e Aplicações* edited by Editora Livraria da Física, São Paulo. It contains an introduction to the fractional calculus whose fourth cover was written by our early deceased friend and prof. José Antônio Tenreiro Machado, unfortunately.
- 2015 – Caputo and Fabrizio → Non-singular kernel. In this year, an article was published in which the so-called non-singular fractional calculus was introduced. After it, several derivative operators with non-singular kernel have been introduced. Again, it has been shown that they cannot be considered classical fractional derivatives.
- 2018 – Vanterler and Edmundo → ψ -Hilfer fractional derivative. In his doctoral thesis, José Vanterler da Costa Sousa introduced the so-called ψ -Hilfer fractional derivative and applied it to the study of erythrocyte sedimentation rates. The main results were presented in J. Vanterler da C. Sousa and E. Capelas de Oliveira, Commun. Nonlinear Sci. Numer. Simulat., **60**, 72-91. The paper was a top three of the journal in that year.
- 2018 – Oliveira and Oliveira → Hilfer-Katugampola derivative. In her doctoral thesis, Daniela dos Santos de Oliveira studied several fractional derivatives and introduced the so-called Hilfer-Katugampola fractional derivative. The work gave rise to D. S. Oliveira and E. Capelas de Oliveira, Comput. Appl. Math., **37**, 3672-3690.
- 2019 – Adrián and Edmundo → Cálculo fraccional. During his PhD, Adrián managed to write what became the first book on fractional calculus written in Spanish. Adrián Ricardo Gómez Plata y Edmundo Capelas de Oliveira, *Introducción al cálculo fraccional*, Editorial Neogranadina, Bogotá. It provides a short introduction to fractional calculus and presents several simple applications.
- 2019 – Teodoro, Tenreiro Machado and Oliveira → A review. During her doctoral thesis, Graziane studied a possible classification of fractional derivatives and operators that resulted in the paper G. Sales Teodoro, J. A. Tenreiro Machado and E. Capelas de Oliveira, *A review of definitions of fractional derivatives and other operators*, J. Comput. Phys., **388**, 195-208.
- 2019 – Edmundo → Solved and proposed exercises. After many years studying fractional calculus and collecting problems and exercises, I published the book *Solved Exercises in Fractional Calculus*, Studies in Systems, Decision and Control 240, Springer Nature Switzerland, Cham.
- 2020 – Considering the existence of several papers and books on fractional derivatives with non-singular kernel, we submitted for publication the article E. Capelas de Oliveira, S. Jarosz, and J. Vaz Jr., *On the mistake in defining fractional derivative using a non-singular kernel*, arXiv:1912.04422v3. Unfortunately, after an intense exchange of correspondence with the

referees and consultation with more experienced researchers, the work, although correct, was refused. We were offered to submit to a paid journal, but we declined. The article is now available on ArXiv. For us bad derivatives are not welcome. I would mention two papers, published that same year, in which this subject is discussed: K. Diethelm, R. Garrappa, A. Giusti, and M. Stynes, *Why fractional derivatives with non-singular kernel should not be used*, *Fract. Calc. & Appl. Anal.*, <https://doi.org/10.1515/fca-2020-0032>; A. Hanyga, *A comment on a controversial issue: A generalized fractional derivative cannot have a regular kernel*, *Frac. Calc. Appl. Anal.*, **23**, 211-223.

- 2021 – Unfortunately, the pandemic came to disrupt many works and contributions. So, for this year, I mention only four works. Two of them can lead to different areas of study, from the point of view of physics and mathematics, and are cited because they are part of our research in the recent past and have contributions that advance in this study topic. First, from the point of view of physics, J. Dong and Y. Lu, *Infinite wall in the fractional quantum mechanics*, *J. Math. Phys.* **62**, 032104. Second, from the point of view of mathematics, J. Boudabsa and T. Simon, *Some properties of the Kilbas-Saigo function*, *Mathematics*, **9**, 217. I would also like to mention two recent papers *in memoriam*: The first was written in honor of the editors who made possible the journal *Fractional Calculus and Applied Analysis*: V. Kiryakova, J. A. Tenreiro Machado, Y. Luchko, *In memory of the honorary founding editors behind the FCAA success story*, *Fract. Calc. Appl. Anal.*, **24**, 641-665. The second, in memory of our friend and enthusiast of fractional calculus: Carla M. A. Pinto, António M. Lopes, and Alexandra M. S. F. Galhano, *In memory of Professor José António Tenreiro Machado (1957-2021)*, *Obituary*, *Nonlinear Dynamics*, <https://doi.org/10.1007/s11071-021-07162-z>.
- 2021 – E. Capelas de Oliveira, *Encounter with fractional calculus*, (in Portuguese), *Revista de Matemática Aplicada e Interdisciplinar* (Editorial). In this editorial I recall my encounter with fractional calculus, from my admission at the university up to my retirement.
- 2022 – II Brazilian Symposium on Fractional Calculus, which will be presented and commented later.

3 Developments in Campinas, Brazil

In this second part I return to year 2005, from which I present graduate students who have been under my guidance or co-orientation, as well as their respective fruits, students formed by former students, now teachers.

- 2005 – Vinícius Alegreti Meza, Rubens Figueiredo Camargo and Edmundo Capelas de Oliveira. This year, the studies of fractional calculus began at Campinas. I compare this with the *equilateral triangle*.
- 2008 – Danilo Castro Resende presents his master of science dissertation entitled *On the Mittag-Leffler functions*, in Portuguese. Since 2019 he is a professor at UEMA, Universidade Estadual do Maranhão, São Luis do Maranhão, Brazil.
- 2009 – Rubens de Figueiredo Camargo presents his doctoral thesis entitled *Fractional Calculus and Applications*, in Portuguese. The first doctoral thesis in fractional calculus under

my guidance and co-oriented by prof. Ary Orozimbo Chiacchio, Department of Mathematics, Institute of Mathematics, Statistical and Scientific Computing, Unicamp. The thesis was awarded honorable mention by the Brazilian Society of Applied and Computational Mathematics. We are now four, Rubens de Figureiredo Camargo, Edmundo Capelas de Oliveira, Ary Orozimbo Chiacchio and Jayme Vaz Jr. A comparison with the *square*. Since 2009, Rubens de Figureiredo Camargo is professor at Unesp, Universidade Estadual Paulista, in Bauru, Brazil, and guided more than twenty scientific initiations and several master of science and doctoral thesis: Arianne Vellasco Gomes (also a professor), Najla Varalta, Robinson Tavoni, Vinícius Machado Martinez, Lislaine Cristina Cardoso, Lucas Kenjy Bezaglia Kuroda, Micaeli Mendola Theodoro, Milena Rodrigues Maciel, Wanderley Ferreira.

- 2010 – Heron Silva Oliveira presents his master of science dissertation entitled *Introduction to arbitrary order calculus*, in Portuguese. Since 2013, he is an assistant professor at Universidade Federal do Maranhão, Imperatriz, Brazil.
- 2011 – Félix Silva Costa presents his doctoral thesis entitled *Fox's H function and its applications in fractional calculus*, in Portuguese. Since 2011, Félix Silva Costa is professor at UEMA, Universidade Estadual do Maranhão, in São Luis do Maranhão, Brazil, and guided several scientific initiations and master of science: Altenize dos Santos Cordeiro Oliveira, Marta Raquel Araujo Pereira, Hugo Pinheiro, Ítalo José Lima de Sousa, José Adson Reis Santos, Lucas Vinícius da Silva.
- 2014 – Eliana Contharteze Grigoletto presents her doctoral thesis entitled *Fractional differential equations and the Mittag-Leffler functions*, in Portuguese. Since 2015, she is professor at Unesp, Universidade Estadual Paulista, in Botucatu, Brazil, and guided several scientific initiations: Luiz Carlos Silva Santana, Pedro Henrique de Oliveira Garcia, Marina Paolacci Carunchio, Larissa Silva Zambrana Moraes, Samuel Ferreira Batista, Giovana Lanza Okada, Daniela Pires de Moraes.
- 2015 – Fábio Grangeiro Rodrigues presents his doctoral thesis entitled *On fractional calculus and solutions of the Bessel equation*, in Portuguese. Unfortunately, he switched from academic fractional calculus to applied statistics.
- 2016 – Júnior César Alves Soares presents his doctoral thesis entitled *Fractional calculus and evolution equations*, in Portuguese. Since 2015, he is professor at Unemat, Universidade Estadual do Mato Grosso, in Barra do Bugres, Brazil, and guided scientific initiations and masters of science: Diana de Almeida dos Santos, Luanderson Nascimento dos Santos, Ynam Fernando Fernandes Xavier, Lecivaldo Lima da Silva.
- 2016 – Adrián Ricardo Gómez Plata presents his doctoral thesis entitled *Nonlinear fractional differential equations*, in Portuguese. Since 2016, he is professor at Universidad Militar Nueva Granada, Colombia and guided two doctoral thesis: Ronald Gentil Rodriguez and Fabio Sora.
- 2017 – Ester Cristina Fontes de Aquino Rosa presents her doctoral thesis entitled *Fractional kinetic relaxation functions*, in Portuguese. Today is professor and research at Federal Institute in Campinas, Brazil.
- 2018 – Daniela dos Santos de Oliveira present her master science dissertation (2014) entitled *Fractional derivatives and the Mittag-Leffler functions*, in Portuguese (2014) and her

doctoral thesis entitled *Fractional derivatives: Generalizations*. Since 2018 is professor and research at Universidade Tecnológica Federal do Paraná, Guarapuava, Brazil. Pos-Doctoral at Universidade Estadual de Campinas.

- 2018 – José Vanterler da Costa Sousa presents his doctoral thesis entitled *Time-fractional diffusion equation (Erythrocyte sedimentation rate)*, in Portuguese. The thesis was awarded as the best thesis of the year, Imecc, Institute of Mathematics, Statistic and Scientific Computation. Since 2020 is professor at Universidade Federal do ABC, Santo André, Brazil. Pos-Doctoral at Universidade Estadual de Campinas and guided master of science and doctoral thesis, Martha Aurora Parra Pulido (Awarded as the best master science dissertation of the year, Imecc), Priscila Santos Ramos, Karla Katherine Barboza de Lima (co-oriented), Jorge Feres Junior, Tomy Felixon.
- 2019 – Graziane Sales Teodoro presented her master of science dissertation entitled *Mittag-Leffler functions and fractional calculus*, in Portuguese (2014) and her doctoral thesis entitled *Fractional derivatives: types and validity criteria*. Since 2019, she is professor and researcher at Universidade Federal de Lavras, Lavras, Brazil.
- 2020 – Karla Katherine Barboza de Lima presents her doctoral thesis entitled *Ulam-like stabilities of solutions of impulsive ψ -Hilfer fractional differential equations*, in Portuguese, co-oriented by prof. José Vanterler da Costa Sousa. Since 2015, she is professor and researcher at Universidade Federal da Grande Dourados, Dourados, Brazil.
- 2021 – Stefânia Jarosz, oriented by prof. Jayme Vaz Jr., presents her master of science dissertation entitled *Fractional Schrödinger equation for delta potentials* and her doctoral thesis entitled *Fractional quantum mechanics studies*, both in the Departamento de Matemática Aplicada, Imecc, Unicamp.
- 2022 – José Ivélton Siqueira Lustosa (in progress) will soon present the works of part of his doctoral thesis, entitled (provisional title) *Fractional calculus applied to evaluate stress concentration and shear effects in simply supported beams*, in Portuguese, oriented by prof. Flávio de Campos Bannwart, Faculdade de Engenharia Mecânica, Unicamp and co-oriented by prof. E. Capelas de Oliveira. Presently, José Ivélton is a high school math teacher at Instituto Federal da Paraíba.
- 2022 – Silas de Sá Cavalcanti Mello (in progress), master of science dissertation in Applied Mathematics at Departamento de Matemática Aplicada, Imecc, Unicamp. He is presently a high school math teacher.
- 2022 – Maria Elismara de Sousa Lima (in progress), doctoral thesis in Applied Mathematics at Departamento de Matemática Aplicada, Imecc, Unicamp, co-oriented by prof. Arlúcio da Cruz Viana, Universidade Federal de Sergipe, Aracaju, Brazil.
- 2022 – Renata Piva Gomes (in progress), master of science dissertation in Applied Mathematics at Departamento de Matemática Aplicada, Imecc, Unicamp. A comparison with the *regular hexagon*.

3.1 Fractional tree

In this subsection I present what I called a fractional tree, a simple tribute to my former advisees, by means of three figures, an equilateral triangle with a square, a hexagon and a dodecagon.

In Figure 1 we have three components, arranged as in an equilateral triangle, Vinícius, Rubens and Edmundo (2005), representing the start of fractional calculus at Unicamp, Campinas, together with a square (this one not drawn), Rubens, Edmundo, Ary and Jayme (2009) first doctoral thesis presented by Rubens Figueiredo Camargo, guided by Edmundo Capelas de Oliveira and co-oriented by Ary Orozimbo Chiacchio. Note that it is shaped like a watch with a fixed minute hand and a moving hour hand. Here, the hours represent the years.

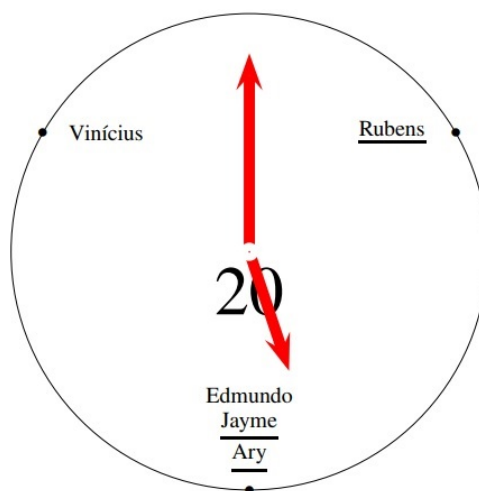


Figure 1: Equilateral triangle and square.

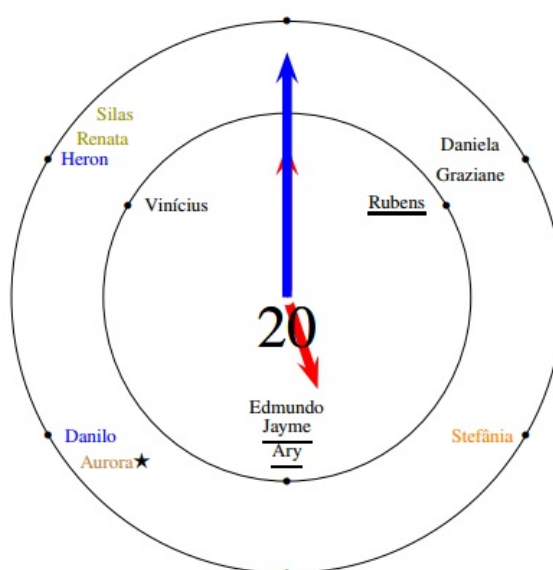


Figure 2: Regular hexagon. Master of science dissertations.

In Figure 2 we have Danilo Castro Rozendo, Heron Silva Oliveira, Daniela dos Santos de Oliveira and Graziane Sales Teodoro, who completed their master of science dissertations; Silas de Sá Cavalcanti Melo and Renata Piva Gomes (in progress) who are presently writing their master of science dissertations, both oriented by me along several years: my regular hexagon. Also, Martha Aurora Parra Pulido, oriented by prof. José Vanterler da Costa Sousa and co-oriented by me, and Stefânia Jarosz, oriented by prof. Jayme Vaz Jr., both masters of science.

I turn now to the works of my former students, who are today my collaborators and friends!

After the equilateral triangle, the square and the regular hexagon we come to the regular dodecagon of Figure 3, depicting the twelve doctoral thesis oriented by me: Rubens de Figueiredo Camargo (co-oriented by prof. Ary Orozimbo Chiacchio), Félix Silva Costa, Eliana Contharteze Grigoletto, Fábio Grangeiro Rodrigues, Júnior César Alves Soares, Adrián Ricardo Gómez Plata, Ester Cristina Fontes de Aquino Rosa, Daniela dos Santos de Oliveira, José Vanterler da Costa Sousa, Graziane Sales Teodoro, Karla Katherine Barboza de Lima (co-oriented by prof. José Vanterler da Costa Sousa) and Maria Elismara de Sousa Lima (co-oriented by prof. Arlúcio da Cruz Viana).

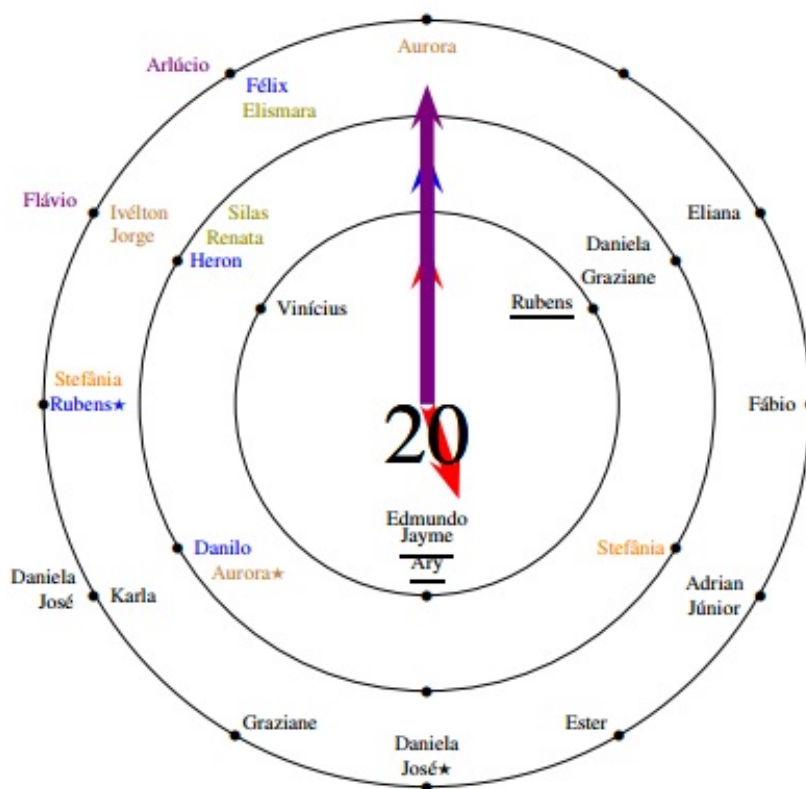


Figure 3: Regular dodecagon. Doctoral thesis.

We also have in Figure 3, José Ivélton Siqueira Lustosa (doctoral thesis in progress), oriented by prof. Flávio de Campos Bannwart and co-oriented by me; Jorge Feres Junior and Martha Aurora Parra Pulido, both with doctoral theses in progress, oriented by prof. José Vanterler da Costa Sousa and co-oriented by me. And out of the regular dodecagon we have Daniela dos Santos de Oliveira and José Vanterler da Costa Sousa, both supervised by me in a post-doctoral program.

I conclude this section recalling that the next regular polygon that fills the space is the icosagon. I leave this task to the professors honored here. Good work!

4 II Brazilian Symposium

I describe here what was the II Brazilian symposium on fractional calculus, highlighting the collaborators, invited speakers and participants.

The II Brazilian Symposium on Fractional Calculus was an online event and took place from 17th to 21st January, 2022. It had two committees:

Scientific committee. The scientific committee was composed of the professors: José Vanterler da Costa Sousa, Daniela dos Santos de Oliveira, Graziane Sales Teodoro, Rubens de Figueiredo Camargo, Edmundo Capelas de Oliveira, César Torres Ledesma and Kishor D. Kucche.

Organizing committee. The organizing committee was composed of the professors: José Vanterler da Costa Sousa (Coordinator) UFABC, Edmundo Capelas de Oliveira, UNICAMP, Rubens de Figueiredo Camargo, UNESP, Adrián Ricardo Gómez Plata, UMNG, Ester Cristina Fontes de Aquino Rosa, IFC, Daniela dos Santos de Oliveira, UTFPR, Eliana Contharteze Grigoletto, UNESP, Félix Silva Costa, UEMA, Gastão Silves Ferreira Frederico, UFC, Graziane Sales Teodoro, UFLA, Júnior Cesar Alves Soares, UNEMAT, Karla Katerine Barboza de Lima, UFGD and Steffânia Jarosz, UNICAMP.

The II Brazilian Symposium on Fractional Calculus, an online event, took place from the 17th to the 21st of January, featured: 9 plenaries, 2 mini-courses, 23 oral presentations, 2 tribute presentations, several papers to be published in *C.Q.D. – Revista Eletrônica Paulista de Matemática* and more than a hundred participants among them students, professors, and researchers.

Plenary lectures. Prof. Manuel Duarte Ortigueira, a representative of the land of Camões and Eça, from the School of Science and Technology of UNL, delivered the opening lecture talking about his work and presenting some challenges and/or controversies around fractional derivatives. Other lecturers were Prof. Richard L. Magin, Department of Bioengineering, University of Chicago Illinois; Prof. Ervin K. Lenzi, Department of Physics, State University of Ponta Grossa, Brazil; Prof. Delfin F. M. Torres (representative of the land of Quental and Camilo, gratitude), Mathematics Department, University of Aveiro, Portugal; Prof. Francesco Mainardi (80 years in december 2022), Free Professor of Mathematical Physics, University of Bologna, Italy; Prof. Roberto Garrappa, University of Bari, Italy; Prof. Carla M. A. Pinto (representative of the land of Herculano and Pessoa), School of Engineering, Polytechnic of Porto, Portugal and Prof. César E. T. Ledesma, Mathematics Department, National University of Trujillo, Peru. Prof. E. Capelas de Oliveira, Unicamp, presented this closing lecture, talking about his work and his former advisees.

Mini-courses. Prof. Stefânia Jarosz taught an introductory mini-course to fractional calculus, *Introduction to fractional calculus*, aimed at undergraduate students and those interested in the subject; and Prof. Gastão Silves Ferreira Frederico, taught a specialized mini-course, *Fractional calculus of variations and optimal control*, aimed at graduate students and researchers.

Tribute presentation. There were two special tribute sessions. First, the *Brazilian Contribution in Fractional Calculus Research*, in which some of the organizers honored prof. E. Capelas de Oliveira; second, the session *In memory of Professor José Antônio Tenreiro Machado*, in which some of the organizers and researchers spoke in honor of late Prof. José Antônio Tenreiro Machado, who had left us on October, 2021.

5 Conclusion

I conclude by expressing my sincerest thanks to the organizers for all the support and promptness in fixing the problems that appeared during this closing lecture.

Gratitude always! Special thanks to my collaborators, ex-students or not, today my friends, who contributed directly or indirectly throughout this ‘fractional’ period.

In time, this lecture is dedicated to Prof. Dr. J. A. Tenreiro Machado, a collaborator, friend, and a true ‘fractional enthusiast’ who left us too soon. *Requiescat in pace*, “May the earth rest lightly on you!”

A question left for everyone. What to expect from fractional calculus 50 years from now?

Thank you!! I hope to see you all at the III Brazilian Symposium on Fractional Calculus, soon.

6 References

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