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Numerical implementation of Mittag-Leffler function: a revision study

Abstract

This work presents a review of an algorithm to calculate the Mittag-Leffler function. In order to do it, we follow the definition of the Mittag-Leffler function in Refs. (GORENFLO; LOUTCHKO; LUCHKO, 2002; DIETHELM et al., 2005) and discuss some of its properties. Then, we revise the numerical algorithm in Ref. (DIETHELM et al., 2005) and plot some cases of Mittag-Leffler function performed. In addition, we discuss the accuracy and convergence of the presented algorithm.

Keywords: Fractional Calculus. Mittag-Leffler. Numerical Method

Eberth de Almeida Correa
Ciências Naturais Aplicadas
Universidade de Brasília, Campus do Gama
eberth@unb.br

Leandro Xavier Cardoso
Ciências Naturais Aplicadas
Universidade de Brasília, Campus do Gama
fiseandro@gmail.com

Tais Calliero Tognetti
Ciências Naturais Aplicadas
Universidade de Brasília, Campus do Gama
tais.calliero@gmail.com

Ronni Geraldo Gomes Amorim
Ciências Naturais Aplicadas
Universidade de Brasília, Campus do Gama
ronniamorim@gmail.com

1 Introduction

The Mittag-Leffler function was introduced in 1903 when G. Mittag-Leffler published a paper about the problem of Laplace-Abel integrals (MITTAG-LEFFLER, 1903, 1903). Further, Wiman extended the concept by introducing the Generalized Mittag-Leffler function (WIMAN, 1905). However, the Mittag-Leffler function became well known after the 1970s when fractional calculus received more attention from mathematicians. In this context, many problems provide a more accurate description of reality when described in terms of fractional order differential equations. In particular, we cite the time-dependent phenomena, in which the fractional derivatives describe memory and hereditary effects in an excellent way (PODLUBNY, 1999). In addition, we emphasize that the impulse response of fractional linear systems can be expressed in terms of Mittag-Leffler function (MAGIN et al., 2011). Finally, the Mittag-Leffler function is one of the most important tools in fractional calculus because it appears in the solutions of some fractional order differential equations.

In this perspective, the computation of the Mittag-Leffler function is not trivial, except when we consider small values of the argument. In general, numerical algorithms for computation of the Mittag-Leffler function present problems of accuracy and convergence. In this sense, we need an efficient algorithm to do it. Then, in this paper, we address this problem and present an efficient numerical algorithm developed by Gorenflo for accurately computing the Mittag-Leffler function. Further, we apply the proposed numerical methodology presenting some figures to compare our calculations to some results from literature. Finally, the results show the convergence power of the algorithm.

The paper is organized as follows. Section 2 introduces the Mittag-Leffler and Generalized Mittag-Leffler function and their properties. Section 3 address the revised algorithm for computing the Mittag-Leffler function. In Section 4, we apply the algorithm. Finally, Section 5 draws the main conclusions.

2 Mittag-Leffler function

This section presents the Mittag-Leffler function and address its main properties.

Magnus G. Mittag-Leffler introduced in 1903 (MITTAG-LEFFLER, 1905) the Mittag-Leffler function, which is one of the most relevant tools to fractional calculus. The Mittag-Leffler function is an extension of the exponential function to arbitrary complex numbers α such as $\mathrm{Re}\{\alpha\} > 0$, and defined as (OLDHAM; SPANIER, 1974; HAUBOLD; MATHAI; SAXENA, 2011)

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}, \tag{1}$$

where $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx$, $\mathrm{Re}\{z\} > 0$, $z \in \mathbb{C}$, denotes the Gamma function.

It is easy to show that the Mittag-Leffler function for some integer values of α is given by (HAUBOLD; MATHAI; SAXENA, 2011; HERRMANN, 2011)

$$E_0(z) = \frac{1}{1-z}, \qquad |z| < 1 \tag{2}$$

$$E_1(z) = e^z \tag{3}$$

$$E_2(z) = \cosh(\sqrt{z}). \tag{4}$$

We can define the Generalized Mittag-Leffler function by (OLDHAM; SPANIER, 1974; HAUBOLD; MATHAI; SAXENA, 2011)

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \tag{5}$$

$\text{Re}\{\alpha\} > 0$, $\text{Re}\{\beta\} > 0$. This means that $E_{\alpha,1}(z) = E_\alpha(z)$.

In the study of the analytical properties of the generalized Mittag-Leffler function one can resort to the integral representations in the complex plane(OLDHAM; SPANIER, 1974; HAUBOLD; MATHAI; SAXENA, 2011)

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i} \oint_C \frac{t^{\alpha-\beta} e^t}{t^\alpha - z} dt, \tag{6}$$

where $\text{Re}\{\alpha\} > 0$, $\text{Re}\{\beta\} > 0$ and the contour C starts and ends at infinite and circles around the singularities and branch points of the integrand.

Below are some examples of functions written in terms of Generalized Mittag-Leffler function (HAUBOLD; MATHAI; SAXENA, 2011; HERRMANN, 2011)

$$\begin{aligned} E_{1,2}(z) &= \frac{e^z - 1}{z} && (7) \\ E_{2,1}(-z^2) &= \cos(z) && (8) \\ E_{2,2}(-z^2) &= \frac{\sin(z)}{z}. && (9) \end{aligned}$$

In the sequence, we present useful properties of Generalized Mittag-Leffler function (HAUBOLD; MATHAI; SAXENA, 2011; HERRMANN, 2011)

$$\begin{aligned} E_{\alpha,\beta}(z) &= zE_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)} && (10) \\ E_{\alpha,\beta}(z) &= \beta E_{\alpha,\beta+1}(z) + \alpha z \frac{d}{dz} E_{\alpha,\beta+1}(z) && (11) \end{aligned}$$

The properties presented here will be useful in understanding the algorithm that will be discussed in the next section. Given the relevance of the Mittag-Leffler function in the framework of fractional calculus, especially in the resolution of fractional differential equations, developing strategies to calculate it accurately is essential for future developments.

For convenience, we follow the definitions in Refs. (GORENFLO; LOUTCHKO; LUCHKO, 2002; DIETHELM et al., 2005), which are only the Eq. (5) with the change of variable $t^\alpha = \lambda$ and a contour C dividing the complex plane into two regions:

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi\alpha i} \int_{\gamma(\rho;\phi)} \frac{e^{z^{1/\alpha}} \lambda^{(1-\beta)/\alpha}}{\lambda - z} d\lambda, \; z \in G^{(-)}(\rho;\phi), \tag{12}$$

and

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} e^{z^{1/\alpha}} + \frac{1}{2\pi\alpha i} \int_{\gamma(\rho;\phi)} \frac{e^{z^{1/\alpha}} \lambda^{(1-\beta)/\alpha}}{\lambda - z} d\lambda, \; z \in G^{(+)}(\rho;\phi). \tag{13}$$

where $\gamma(\rho;\phi)$ is a contour in the complex λ-plane obeying the following requirements:

(a) the ray $\arg\lambda = -\phi$, $|\lambda| \geq \rho$ with $\rho > 0$, $0 < \phi \leq \pi$;
(b) the arc $-\phi \leq \arg\lambda \leq \phi$ from the circumference $|\lambda| = \rho$;
(c) the ray $\arg\lambda = \phi$, $|\lambda| \geq \rho$.

The domains $G^{(+)}(\rho;\phi)$ and $G^{(-)}(\rho;\phi)$ are to the right and the left of the contour $\gamma(\rho;\phi)$, respectively, dividing the complex λ-plane into two unbounded parts when $0 < \phi < \pi$. For $\phi = \pi$, $G^{(-)}(\rho;\phi)$ becomes the circle $|\lambda| < \rho$ and, consequently, $G^{(+)}(\rho;\phi)$ will comprise the region $\{\lambda : |\arg\lambda| < \pi,\ |\lambda| > \rho\}$. The integral representations (12) and (13) are such that $0 < \alpha < 2$, β is arbitrary and

$$\frac{\alpha\pi}{2} < \phi \leq \min\{\pi, \alpha\pi\} \tag{14}$$

In Ref. (GORENFLO; LOUTCHKO; LUCHKO, 2002) one can find the following asymptotic expansions(with $|z| \to \infty$) for $0 < \alpha < 2$, β being arbitrary, and ϕ inside the interval (14):

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} e^{z^{1/\alpha}} - \sum_{k=0}^{p} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + O(|z|^{-1-p}),\ \text{for } |\arg z| < \phi, \tag{15}$$

and

$$E_{\alpha,\beta}(z) = -\sum_{k=0}^{p} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + O(|z|^{-1-p}),\ \text{for } \phi \leq |\arg z| \leq \pi, \tag{16}$$

with $p \in \mathbb{N}$. It is also possible to find more suitable formulas obtained from the integral representations (12) and (13) in the particular case for $0 < \alpha \leq 1$, $\beta \in \mathbb{R}$, and $|z| \neq 0$:
1)$|\arg z| > \alpha\pi$

$$E_{\alpha,\beta}(z) = \int_0^\infty K(\alpha,\beta,\chi,z)d\chi, \quad \text{if } \beta < 1 + \alpha \tag{17}$$

$$E_{\alpha,\beta}(z) = \int_\rho^\infty K(\alpha,\beta,\chi,z)d\chi + \int_{-\alpha\pi}^{\alpha\pi} P(\alpha,\beta,\rho,\phi,z)d\phi, \quad \rho > 0,\ \beta \in \mathbb{R} \tag{18}$$

2)$|\arg z| < \alpha\pi$

$$E_{\alpha,\beta}(z) = \int_0^\infty K(\alpha,\beta,\chi,z)d\chi + \frac{1}{\alpha} z^{(1-\beta)/\alpha} e^{z^{1/\alpha}}, \quad \text{if } \beta < 1 + \alpha \tag{19}$$

$$E_{\alpha,\beta}(z) = \int_\rho^\infty K(\alpha,\beta,\chi,z)d\chi + \int_{-\alpha\pi}^{\alpha\pi} P(\alpha,\beta,\rho,\phi,z)d\phi + \frac{1}{\alpha} z^{(1-\beta)/\alpha} e^{z^{1/\alpha}}, \quad 0 < \rho < |z|,\ \beta \in \mathbb{R} \tag{20}$$

3)$|\arg z| = \alpha\pi$

$$E_{\alpha,\beta}(z) = \int_\rho^\infty K(\alpha,\beta,\chi,z)d\chi + \int_{-\alpha\pi}^{\alpha\pi} P(\alpha,\beta,\rho,\phi,z)d\phi, \quad \rho > |z|,\ \beta \in \mathbb{R} \tag{21}$$

with

$$
\begin{aligned}
K(\alpha,\beta,\chi,z) &= \frac{1}{\alpha\pi}\chi^{(1-\beta)/\alpha}\exp\left(-\chi^{1/\alpha}\right)\frac{\chi\sin\left[\pi(1-\beta)\right]-z\sin\left[\pi(1-\beta+\alpha)\right]}{\chi^2-2\chi z\cos\left(\alpha\pi\right)+z^2} && (22)\\
P(\alpha,\beta,\rho,\phi,z) &= \frac{1}{2\alpha\pi}\rho^{1+(1-\beta)/\alpha}\exp\left[\rho^{1/\alpha}\cos\left(\phi/\alpha\right)\right]\frac{\cos\left(\omega\right)+i\sin\left(\omega\right)}{\rho\exp\left(i\phi\right)-z} && (23)\\
\omega &= \phi[1+(1-\beta)/\alpha]+\rho^{1/\alpha}\sin\left(\phi/\alpha\right) && (24)
\end{aligned}
$$

Finally, for $\alpha > 1$ and $\beta \in \mathbb{R}$ one can use the following especial formula:

$$
E_{\alpha,\beta}(z) = \frac{1}{k_0}\sum_{k=0}^{k_0-1} E_{\alpha/k_0,\beta}\left(z^{1/k_0}\exp\left(i2\pi k/k_0\right)\right) \tag{25}
$$

with $k_0 = \lfloor\alpha\rfloor+1$. Other special cases can be found in Refs. (GORENFLO; LOUTCHKO; LUCHKO, 2002; DIETHELM et al., 2005)that are omitted here since the revised algorithm is presented in the next section.

3 Numerical algorithm

The algorithm summarizes the conditions and results of Eqs. (15)-(25). Special attention is deserved to Eq. (21) and its condition of convergence. In fact, the version of algorithm found in Ref. (GORENFLO; LOUTCHKO; LUCHKO, 2002) uses $\rho = (|z|+1)/2$ in Eq. (21), which is not correct. To circumvent this situation we choose $\rho = |z|+1$ to satisfy the correct condition playing an important role when one calculates the derivative of the Mittag-Leffler function as we will see in the next section. We present bellow the revised numerical algorithm from Ref. (DIETHELM et al., 2005) to calculate the Mittag-Leffler function with minor corrections.

Algorithm 1

Require: $\alpha > 0, \beta \in \mathbb{R}, \epsilon > 0, 0 < \zeta < 1, z \in \mathbb{C}$

if $z = 0$ **then**
  $E_{\alpha,\beta}(0) = 1/\Gamma(\beta)$
else if $\alpha = \beta = 1$ **then**
  $E_{1,1}(z) = e^z$
else if $1 < \alpha$ **then**
  $k_0 = \lfloor\alpha\rfloor + 1$
  $E_{\alpha,\beta}(z) = \frac{1}{k_0}\sum_{k=0}^{k_0-1} E_{\alpha/k_0,\beta}\left(z^{1/k_0}\exp\left(i2\pi k/k_0\right)\right)$
else
  if $|z| < \zeta$ **then**
    $k_0 = max\left\{\lceil(1-\beta)/\alpha\rceil, \lceil\ln\left[\epsilon(1-|z|)\right]/\ln\left(|z|\right)\rceil\right\}$
    $E_{\alpha,\beta}(z) = \sum_{k=0}^{k_0} z^k/\Gamma(\beta+\alpha k)$
  else if $|z| < \lfloor 10+5\alpha\rfloor$ **then**

$$\chi_0 = \begin{cases} max\left\{1, 2|z|, [-\ln(\epsilon\pi/6)]^{\alpha}\right\} & \beta \geq 0 \\ max\left\{(|\beta|+1)^{\alpha}, 2|z|, \left[-2\ln\left(\frac{\epsilon\pi}{6(|\beta|+2)(2|\beta|)^{|\beta|}}\right)\right]^{\alpha}\right\} & \beta < 0 \end{cases}$$

$$K(\alpha,\beta,\chi,z) = \frac{1}{\alpha\pi}\chi^{(1-\beta)/\alpha}\exp(-\chi^{1/\alpha})\frac{\chi\sin[\pi(1-\beta)] - z\sin[\pi(1-\beta+\alpha)]}{\chi^2 - 2\chi z\cos(\alpha\pi) + z^2}$$

$$P(\alpha,\beta,\rho,\phi,z) = \frac{1}{2\alpha\pi}\rho^{1+(1-\beta)/\alpha}\exp[\rho^{1/\alpha}\cos(\phi/\alpha)]\frac{\cos(\omega) + i\sin(\omega)}{\rho\exp(i\phi) - z}$$

$$\omega = \phi[1 + (1-\beta)/\alpha] + \rho^{1/\alpha}\sin(\phi/\alpha)$$

if $|\arg z| > \alpha\pi$ **and** $||\arg z| - \alpha\pi| > \epsilon$ **then**

  if $\beta < 1 + \alpha$ **then**

$$E_{\alpha,\beta}(z) = \int_0^{\chi_0} K(\alpha,\beta,\chi,z)d\chi$$

  else

$$E_{\alpha,\beta}(z) = \int_1^{\chi_0} K(\alpha,\beta,\chi,z)d\chi + \int_{-\alpha\pi}^{\alpha\pi} P(\alpha,\beta,1,\phi,z)d\phi$$

  end if

else if $|\arg z| < \alpha\pi$ **and** $||\arg z| - \alpha\pi| > \epsilon$ **then**

  if $\beta < 1 + \alpha$ **then**

$$E_{\alpha,\beta}(z) = \int_0^{\chi_0} K(\alpha,\beta,\chi,z)d\chi + \frac{1}{\alpha}z^{(1-\beta)/\alpha}e^{z^{1/\alpha}}$$

  else

$$E_{\alpha,\beta}(z) = \int_{|z|/2}^{\chi_0} K(\alpha,\beta,\chi,z)d\chi + \int_{-\alpha\pi}^{\alpha\pi} P(\alpha,\beta,|z|/2,\phi,z)d\phi + \frac{1}{\alpha}z^{(1-\beta)/\alpha}e^{z^{1/\alpha}}$$

  end if

else

$$E_{\alpha,\beta}(z) = \int_{|z|+1}^{\chi_0} K(\alpha,\beta,\chi,z)d\chi + \int_{-\alpha\pi}^{\alpha\pi} P(\alpha,\beta,|z|+1,\phi,z)d\phi$$

▷ Modified expression

end if

else

$$a(k) = \begin{cases} 0 & \text{if } \beta - \alpha k \in \{0\} \text{ or } \mathbb{Z}^- \\ 1/\Gamma(\beta - \alpha k) & \text{otherwise} \end{cases}$$

▷ To avoid divergences

$k_0 = \lfloor -\ln(\epsilon)/\ln(|z|) \rfloor$

if $|\arg z| < 3\alpha\pi/4$ **then**

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha}z^{(1-\beta)/\alpha}e^{z^{1/\alpha}} - \sum_{k=0}^{k_0} z^{-k}a(k)$$

else

$E_{\alpha,\beta}(z) = -\sum_{k=0}^{k_0} z^{-k} a(k)$

end if

end if

end if

4 Algorithm tests

To deploy Algorithm 1, we used Python version 3.7.4. For the integrals present in the algorithm we used the adaptive *scipy.integrate.quad* package, which uses a technique from the Fortran library *QUADPACK*. The default absolute error of convergence for all integrals is $1.49{\times}10^{-8}$, which we found to be reasonable for our calculations. To manipulate arrays, object types and calculate some necessary mathematical functions, we resort to the *numpy* package. Special attention must be deserved to the function $P(\alpha, \beta, \rho, \phi, z)$. The reason for that is that this function oscillates around the origin. To get the precision we were looking for we divided the interval of integration($-\alpha\pi \leq \phi \leq \alpha\pi$) into ten subintervals and then applied the adaptive *scipy.integrate.quad*.

Depending on the values of α and if $0 \leq |z| \leq \chi_0$, the $K(\alpha, \beta, \chi, z)$ function can manifest singularities at values of $\chi = |z|$. To circumvent this situation one can divide the interval of integration into two parts: $0 \leq \chi \leq |z| - \delta$ and $|z| + \delta \leq \chi \leq \chi_0$, with $\delta = 2 \times 10^{-4}$ at most of the cases. We found that a better precision can be reached if one uses the *scipy.integrate.quadrature* package with the options *tol=1E-14, rtol=1E-14, maxiter=1000* set.

In the work of Diethelm *et al.*(DIETHELM et al., 2005), the parameters used in Algorithm 1 must be such that ϵ is equal to the machine epsilon and ζ must be closer to 1. For ζ, we used 0.9 for all calculations and $\epsilon = 2.22 \times 10^{-16}$ (the machine epsilon), as suggested in Ref.(DIETHELM et al., 2005). This does not mean one can reach such a precision for every calculation. To illustrate this, we tested the analytical result of Eq. (7) against the Algorithm 1 for several values of z. In this particular case, $\alpha = 1$, $\beta = 2$, and z is a real number. In Python the *numpy* package gives an efficient function to calculate $\arg z$ in the interval $-\pi < \arg z \leq \pi$ using *numpy.angle(z)*. The absolute error values between $E_{1,2}(z)$ and the analytical expression Eq. (7) are summarized in Table 1.

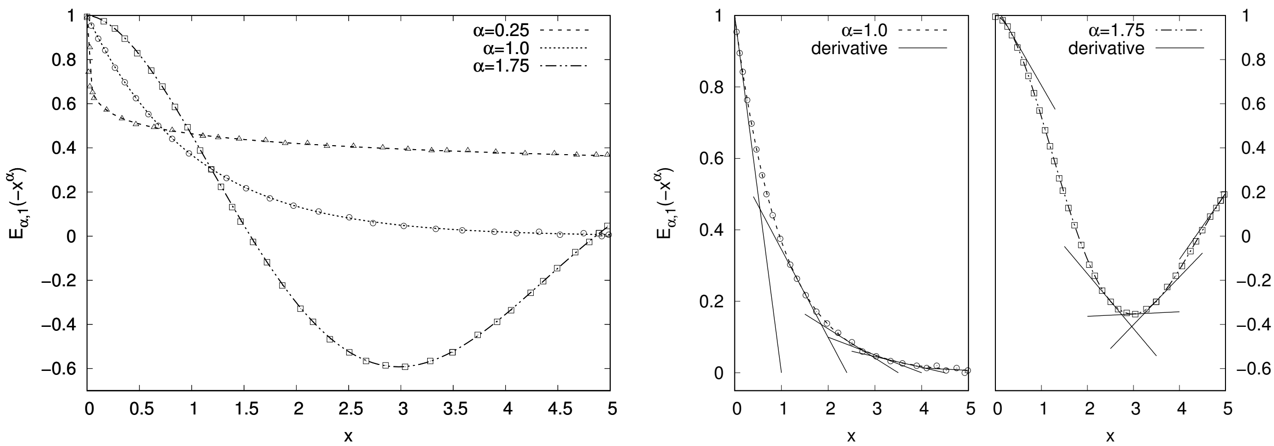
Table 1: Absolute errors calculated for Eq. (7). These results are for $\epsilon = 2.22 \times 10^{-16}$.

Function	z =-15.5	z =-10	z =-5	z =-1	z =1	z =5	z =10	z =25
$E_{1,2}(z)$	$1.20{\times}10^{-8}$	$4.73{\times}10^{-15}$	$2.78{\times}10^{-17}$	$1.11{\times}10^{-16}$	$2.22{\times}10^{-16}$	0.00	0.00	0.00

For values in the interval $-15 < |z| < 15$, we obtained at most $21.3 \times \epsilon$ for the absolute error value. However, for the value $z = -15.5$ we obtained about 1.20×10^{-8} for the absolute error value. This result is due to the relations at the end of Algorithm 1 given by Eqs. (15) and (16). For this particular choice of $\alpha = 1$ and $\beta = 2$, the values of the argument of the Gamma function that appears in the summation, $\Gamma(\beta - \alpha k)$, can assume negative integer values, which is well known that diverges. In Python, this is not a problem in a division, which gives a result of zero in an ordinary case. However, this can be a problem for deployment in other programming languages. For this reason, we defined the function $a(k)$ in Algorithm 1. In Python, it is straightforward to avoid divergences with this function just giving the one-line command:

```
a=lambda x: 0 if x.is_integer()==True and x<0 else 1./gamma(x)
```

where $x \equiv \beta - \alpha k$ and gamma is the *scipy.special.gamma* function. Note that when $|z| \geq 15$, $E_{\alpha,\beta}$ is calculated by Eqs. (15) and (16), but k_0 assumes values like $\lfloor -\ln(\epsilon)/\ln(|z|)\rfloor$, which in our case is about 13. It happens that $\beta - \alpha k$ assumes negative integer values from $k > 1$ in this particular case of α and β. Then, for $k = 1$ the asymptotic behavior of $E_{1,2}$ is inside the convergence radius and converges with $1/z$. That is the reason we obtained such an absolute error value at $|z| = -15.5$, which is calculated through Eq. (16), converging with $1/z$, for negative values. For example, if $|z| = -25$ one gets 5.56×10^{-13}, corroborating our assertion. For positive values, there is no problem and the truncated series converges well.



(a) Mittag-Leffler function for $z = -x^{\alpha}$

(b) The derivatives of $E_{\alpha,1}(-x^{\alpha})$ at some points using the algorithm in Ref. (DIETHELM et al., 2005).

Figure 1: Mittag-Leffler function calculated using Algorithm 1. The open circles, triangles and squares are the data from the Ref. (DIETHELM et al., 2005).

Finally, we used the algorithm presented in Ref. (DIETHELM et al., 2005) to calculate the derivative of $E_{\alpha,\beta}(z)$. We tested this algorithm, which seems to give good results, omitted for brevity. This algorithm uses the values of $E_{\alpha,\beta}(z)$ calculated through Algorithm 1. For this reason, Algorithm 1 must be rigorously tested. Besides small corrections in this algorithm, we also performed numerical tests for the recurrence relations (10) and (11). They were confirmed with the same precision reached for every point in Table 1. In Figures 1(a) and 1(b), we plot function $E_{\alpha,1}(-x^{\alpha})$ for $0 \leq x \leq 5$, $\alpha = 0.25, 1, 1.75$, and the derivatives at some points. In Figure 1(a), we compare our calculations with those obtained by Diethelm *et al.*(DIETHELM et al., 2005). We reproduced their results with very good precision. However, in Figure 1(b), the derivatives for values such that $x > 1.4$, with $\alpha = 1$, were not correct using the algorithm in Ref. (DIETHELM et al., 2005). The reason is the value of ρ calculating the integrals K and P to obtain $E_{\alpha,1}(z)$. As mentioned before, in their original paper, Gorenflo *et al.*(GORENFLO; LOUTCHKO; LUCHKO, 2002) used $\rho = (|z|+1)/2$ in Eq. (21), which is not correct for satisfying the convergence condition. This error propagated towards the revised algorithm in Ref. (DIETHELM et al., 2005). As we suggested through Algorithm 1, the choice $\rho = |z| + 1$ satisfies the correct condition and gives a good convergence in the calculation of $E_{\alpha,1}(z)$, which implies the correct values of the derivatives in Figure 1(b).

5 Conclusion

We have tested the numerical algorithm suggested by Gorenflo *et al.*(GORENFLO; LOUTCHKO; LUCHKO, 2002) and revised by Diethelm *et al.*(DIETHELM et al., 2005). A small error in the parameter ρ for calculating the integrals K and P to obtain $E_{\alpha,\beta}(z)$ was corrected and we tested the algorithm again, obtaining more appropriate results for some analytical relations. In Ref. (DIETHELM et al., 2005), the authors furnish a table with the coefficients of Padé approximates for $E_{\alpha,1}(-x^{\alpha})$, $0 < \alpha < 1$ and $x \in [0.1, 15]$. This is an efficient way to deploy $E_{\alpha,\beta}(z)$, with a huge computational gain for applications in engineering and physics, mainly if one wants to solve partial differential equations through finite element mesh techniques. Nonetheless, these tables must be revised once, for some values of x and α, the wrong condition for calculating the integrals K and P to obtain $E_{\alpha,\beta}(z)$ can cause different values from the correct ones. Indeed, these values may suffer only small changes because most of the contribution for $E_{\alpha,\beta}(z)$ is due to the integral of K, which is not affected severely by this mistake. We intend to test this table with Algorithm 1 in the future.

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