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A Yamazaki-type estimate to a space-time fractional diffusion equation

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Abstract

In this work, we develop the Yamazaki inequality and estimates of the Mittag-Leffler family associated with the space-time fractional diffusion equation in L^p-weak spaces. This result generalizes the inequality given by Yamazaki (2000) for the heat semigroup, which was generalized by Ferreira and Villamizar-Roa (2006) for heat semigroup associated with the fractional Laplacian, and Caicedo *et al* (2021) for the Mittag-Leffler family associated with the time-fractional diffusion equation.

Keywords: Yamazaki-type estimate, fractional diffusion equation, Mittag-Leffler family, decay in L^p spaces.

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1 Introduction

Fractional order diffusion equations describe anomalous diffusion phenomena, which aid in the analysis of systems such as: plasma diffusion, fractal diffusion, anomalous diffusion on liquid surfaces, analysis of heart beat histograms in healthy individuals, among other physical systems (see e.g. (BILER; FUNAKI; WOYCZYNSKI, 1998) and (BONKILE et al., 2018)). In order to understand the following text, we need the following definition which can be found in (KILBAS; SRIVASTAVA; TRUJILLO, 2006).

Definition 1.1 *Let be $\Omega = [a, b]$ $(-\infty < a < b < \infty)$ a finite interval over $\mathbb{R}$. Riemann-Liouville fractional integrals, $I^{\alpha}_{a^+}$ and $I^{\alpha}_{b^-}$ of order $\alpha \in \mathbb{R}$ $(\alpha > 0)$ are given by:*

$$(I^{\alpha}_{a^+} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \ t > a,$$

$$(I^{\alpha}_{b^-} f)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) ds, \ t < b,$$

where $\Gamma(\alpha)$ is the gamma function and $f \in L^1[a, b]$.

Definition 1.2 *Riemann-Liouville fractional derivatives, $D^{\alpha}_{a^+}$ and $D^{\alpha}_{b^-}$ of order $\alpha \in \mathbb{R}$ $(\alpha > 0)$ are given by*

$$(D^{\alpha}_{a^+} f)(t) = \left(\frac{d}{dt}\right)^n (I^{n-\alpha}_{a^+} f)(t), \ (n = [\alpha] + 1, t > a),$$

$$(D^{\alpha}_{b^-} f)(t) = (-1)^n \left(\frac{d}{dt}\right)^n (I^{n-\alpha}_{b^-} f)(t), \ (n = [\alpha] + 1, t < b),$$

where $[\alpha]$ means the whole part of α and $f : I \to \mathbb{R}$. We take $n = \alpha$, if $\alpha \in \mathbb{N}_0$.

In this article, we will cover estimates for $G_{\alpha,\gamma}(t)$, defined by

$$\widehat{G_{\alpha,\gamma}(t)\phi}(\xi) = E_{\alpha}(-t^{\alpha}|\xi|^{2\gamma})\widehat{\phi}(\xi), \tag{1}$$

which is the Mittag-Leffler family associated with fractional Laplacian of the space-time diffusion equation and initial condition

$$u_t + D^{1-\alpha}_t (-\Delta)^{\gamma} u(t) = 0, \qquad (x, t) \in \mathbb{R}^n \times (0, \infty), \tag{2}$$

$$u(x, 0) = u_0(x), \qquad x \in \mathbb{R}^n, \tag{3}$$

where $\alpha, \gamma \in (0, 1)$, the fractional Laplacian operator is defined, as usual, through the Fourier transform as

$$(\widehat{(-\Delta)^{\gamma}\varphi})(\xi) = |\xi|^{2\gamma}\widehat{\varphi}(\xi), \tag{4}$$

with the convention

$$\widehat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x) dx, \tag{5}$$

and $D^{\beta}_t \varphi$ denotes the fractional derivative of φ of order $\beta > 0$ in the sense of Riemann-Liouville (see **Definition1.2**).

Anomalous diffusion can be characterized both by admitting jumps of the type *Levy flights*, mathematically represented by the fractional Laplacian, as well as *long rests* described by the fractional time–derivative. In this case, the appropriate equation, according to Schneider and Wyss (SCHNEIDER; WYSS, 1989) and Metzler and Klafter (METZLER; KLAFTER, 2000), is given by equation (2).

First, we present some concepts in the spaces L^p-weak, $p > 1$, to estimate $G_{\alpha,\gamma}(t)$, the Mittag-Leffler family and its derivatives, associated with equation (2). So we can state and prove Yamazaki's estimate for $G_{\alpha,\gamma}(t)$ in Section 3, which is our main goal.

Next we will determine $G_{\alpha,\gamma}(t)$ through formal calculations. In equation (2), using the definition of fractional derivative (see **Definition 1.2**), we have

$$u_t + \partial_t \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (-\Delta)^\gamma u(s) ds = 0, \qquad (x,t) \in \mathbb{R}^n \times (0,\infty), \tag{6}$$

where $0 < \alpha, \gamma < 1$.

Now, we apply a procedure similar as in the Duhamel's principle, to obtain a solution, using the Fourier transform and then its inverse, performing formal calculations and assuming that $(-\Delta)^\gamma u$ is continuous. Integrating equation (6) with respect to t,

$$u + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (-\Delta)^\gamma u(s) ds = u_0. \tag{7}$$

Applying the Fourier transform, it follows that

$$\widehat{u} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \widehat{(-\Delta)^\gamma u}(\xi, s) ds = \widehat{u_0}, \tag{8}$$

using the definition of fractional Laplacian, we have

$$\widehat{u} + |\xi|^{2\gamma} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \widehat{u}(\xi, s) ds = \widehat{u_0}. \tag{9}$$

Applying the Laplace transform to equation (9), it follows that

$$\tilde{\widehat{u}} \left[1 + \frac{|\xi|^{2\gamma}}{s^\alpha} \right] = \frac{\widehat{u_0}}{s}.$$

Thus,

$$\tilde{\widehat{u}} = \frac{s^{\alpha-1}}{s^\alpha + |\xi|^{2\gamma}} \widehat{u_0}. \tag{10}$$

Applying the inverse Laplace transform to equation (10), it follows that

$$\widehat{u}(s) = E_\alpha(-|\xi|^{2\gamma} t^\alpha) \widehat{u_0}, \tag{11}$$

where $E_\alpha(\cdot)$ is the classical Mittag-Leffler function. This implies that

$$u(x,t) = G_{\alpha,\gamma}(t) u_0 \tag{12}$$

where,

$$\widehat{G_{\alpha,\gamma}(t)\phi}(\xi) = E_\alpha(-t^\alpha |\xi|^{2\gamma}) \widehat{\phi}(\xi). \tag{13}$$

In appendix E of (MAINARDI, 2010), we can see that the Mittag-Leffler function can be written as an integral. Thus,

$$E_\alpha(-t^\alpha|\xi|^{2\gamma}) = \int_0^\infty e^{-\eta t^\alpha|\xi|^{2\gamma}} M_\alpha(\eta)d\eta. \tag{14}$$

Here M_α denotes the Wright function which is non-negative, $M_\alpha(t) \geq 0$, satisfying

$$\int_0^\infty M_\alpha(s)s^r ds = \frac{\Gamma(1+r)}{\Gamma(1+\alpha r)}, \tag{15}$$

whenever $r > -1$ (See appendix F of (MAINARDI, 2010)). The calculation above shows that

$$G_{\alpha,\gamma}(t) = \int_0^\infty M_\alpha(\eta)G_\gamma(\eta t^\alpha)d\eta, \tag{16}$$

where $G_\gamma(t)$ is the semigroup associated with the fractional Laplacian $(-\Delta)^\gamma$, that is,

$$\widehat{G_\gamma(t)\phi}(\xi) = e^{-|\xi|^{2\gamma}t}\widehat{\phi}(\xi). \tag{17}$$

So $G_{\alpha,\gamma}(t)u_0$ checks equation (2) (see (BAZHLEKOVA; BAZHLEKOV, 2019)). Our sake is to estimate equation (16) in L^p-weak spaces.

The Yamazaki inequality is given by (see (YAMAZAKI, 2000)): If $1 < p < q < \infty$, there is a constant $C > 0$ such that

$$\int_0^\infty t^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-1}\|G(t)\phi\|_{(q,1)}ds \leq C\|\phi\|_{(p,1)}, \tag{18}$$

for each $\phi \in L^{(p,1)}(\mathbb{R}^n)$ and $G(t)$ is the heat semigroup, defined by (17), with $\gamma = 1$. Thus, Theorem 3.2 that we will prove in the Section 3, is an adaptation of a technical Lemma generalized by Ferreira and Villamizar-Roa, given by (see (FERREIRA; VILLAMIZAR-ROA, 2006)): Let $0 < \gamma < \infty$ and $1 < p < q < \infty$, then

$$\int_0^\infty t^{\frac{n}{2\gamma}\left(\frac{1}{p}-\frac{1}{q}\right)-1}\|G_\gamma(t)\phi\|_{(q,1)}ds \leq C\|\phi\|_{(p,1)}. \tag{19}$$

A similar result is shown by the authors Caicedo *et al* (see (CAICEDO et al., 2021)), who consider a fractional version of Yamazaki inequality, given by: Let $1 < p < q < \infty$ be such that $\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right) < 1$. There exists a constant $C > 0$ such that

$$\int_0^\infty t^{\frac{n\alpha}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-1}\|G_\alpha(t)\phi\|_{(q,1)}ds \leq C\|\phi\|_{(p,1)}, \tag{20}$$

for each $\phi \in L^{(p,1)}(\mathbb{R}^n)$, where $G_\alpha(t) = \int_0^\infty M_\alpha(\eta)G(\eta t^\alpha)d\eta$ and $G(t)$ is the heat semigroup. Next, we state the main result of this paper: Let $1 < p < r < \infty$ such that $0 < \frac{n}{2\gamma}\left(\frac{1}{p}-\frac{1}{r}\right)+\frac{1}{2\gamma} < 1$. There is $C > 0$ such that

$$\int_0^\infty t^{\frac{n\alpha}{2\gamma}\left(\frac{1}{p}-\frac{1}{r}\right)+\frac{\alpha}{2\gamma}-1}\left\|\frac{\partial}{\partial x_i}G_{\alpha,\gamma}(t)\phi\right\|_{(r,1)} ds \leq C\|\phi\|_{(p,1)}, \tag{21}$$

$\forall\ \phi \in L^{(p,1)}(\mathbb{R}^n)$.

An application of such estimates is given when one seeks to prove the existence of the solution for the differential equation

$$u_t + D_t^{1-\alpha}(-\Delta)^\gamma u(s)ds = a \cdot \nabla(u^\rho), \tag{22}$$

where $\rho > 1$, $a \in \mathbb{R}^n$, which can be motivated due to the great physical applicability of the space fractional Burgers equation

$$u_t = \nu(-\Delta)^{\beta/2}u - a \cdot \nabla(u^r), \tag{23}$$

where $r > 1$, $x \in \mathbb{R}^d$ $d = 1, 2, \cdots, t \geq 0$, $u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$, $\beta \in (0, 2]$ and $a \in \mathbb{R}^d$ is a fixed vetor. It has been used to model a variety of physical phenomena. Among them we can mention the growth of molecular interfaces, where shock creation is an important factor. We can also cite statistical properties of solutions to the Burgers equation with initial random data with intriguing connections for probability theory, stochastic partial differential equations, chaos propagation and numerical simulations (See (BILER; FUNAKI; WOYCZYNSKI, 1998)).

2 Preliminary

In this section, we will recall definitions and basic results on Lorentz spaces needed to prove the Yamazaki-type estimate for $G_{\alpha,\gamma}(t)$.

Next we will define the L^p–weak spaces (or Marcinkiewicz spaces, see (BERGH; LÖFSTRÖM, 1976)). For this, consider a measurable function $f : \mathbb{R}^n \to \mathbb{R}$. We define the f^* rearrangement function by

$$f^*(t) = \inf\{s > 0 : m(\{x \in R^n : |f(x)| > s\}) \leq t\}, t > 0, \tag{24}$$

and the average function f^{**} by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds, t > 0, \tag{25}$$

where m is the measure $\mathbb{R}^n$-Lebesgue.

Let $0 < p \leqslant \infty, 0 < q \leqslant \infty$. Lorentz space, $L^{(p,q)}(\mathbb{R}^n)$, is the set of all measurable functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, such that $\|f\|^*_{(p,q)} < \infty$, where

$$\|f\|^*_{(p,q)} = \begin{cases} \left[\frac{q}{p}\int_0^\infty \left(t^{\frac{1}{p}}f^*(t)\right)^q \frac{dt}{t}\right]^{\frac{1}{q}}, & \text{if } 0 < p < \infty, 0 < q < \infty, \\ \sup_{t>0} t^{\frac{1}{p}}f^*(t), & \text{if } 0 < p \leqslant \infty, q = \infty. \end{cases} \tag{26}$$

Since $\|f\|^*_{(p,q)}$ does not satisfy the triangular inequality, $L^{(p,q)}$ is metrized with the norm $\|f\|_{(p,q)}$ given by

$$\|f\|_{(p,q)} = \begin{cases} \left(\frac{q}{p}\int_0^\infty \left[t^{\frac{1}{p}}f^{**}(t)\right]^q dt/t\right)^{\frac{1}{q}}, & \text{if } 1 < p < \infty, 1 \leq q < \infty. \\ \sup_{t>0} t^{\frac{1}{p}}f^{**}(t), & \text{if } 1 < p \leq \infty, q = \infty. \end{cases} \tag{27}$$

Functions of this type can be both singular and homogeneous, so that there are more singular functions in Marcinkiewcz than in Lebesgue spaces. Then, studying partial differential equations in L^p - weak spaces allows one to seek self-similar solutions and consider singular initial data at the same time. For example, we have the homogeneous functions of degree $\frac{-n}{p}$ that are in $L^{(p,\infty)}(\mathbb{R}^n)$, and we can quote, $f(x) = |x|^{\frac{-n}{p}}$.

In fact, we have

$$m(\{x \in \mathbb{R}^n : |x|^{-\frac{n}{p}} > s\}) = m(\{x \in \mathbb{R}^n : s^{-\frac{p}{n}} > |x|\}) = V_n(s^{-\frac{p}{n}})^n = V_n s^{-p},$$

where V_n is the volume of an n-dimensional unit ball. Thus,

$$f^*(t) = \inf\{s > 0 : V_n s^{-p} \leq t\} = \inf\{s > 0 : V_n^{1/p} t^{-1/p} \leq s\} = V_n^{1/p} t^{-1/p}.$$

Next,

$$\|f^*\|_{(p,\infty)} = \sup_{t>0} t^{1/p} f^*(t) = \sup_{t>0}(t^{1/p} V_n^{1/p} t^{-1/p}) = V_n^{1/p} < \infty.$$

Example 2.1 *Show that, for $t \in (0, \infty)$, $\|t^{-l_j^{-1}}\|_{(l_j,\infty)} = 1$.*

Proof:
In fact,

$$m(\{t \in \mathbb{R} : |t|^{-l_j^{-1}} > s\}) = m(\{t \in \mathbb{R} : s^{-l_j} > |t|\}) = s^{-l_j}.$$

thus,

$$f^*(t) = \inf\{s > 0 : s^{-l_j} \leq t\} = \inf\{s > 0 : t^{-1/l_j} \leq s\} = t^{-1/l_j}$$

Next,

$$\|f^*\|_{(l_j,\infty)} = \sup_{t>0} t^{1/l_j} f^*(t) = \sup_{t>0}(t^{1/l_j} t^{-1/l_j}) = 1$$

The generalization of the Hölder inequality is given by the following proposition.

Proposition 2.2 (Theorem 3.4, (O'NEIL, 1963) – Generalized Holder's inequality) *Let $1 < p_1, p_2 < \infty$. Let $f \in L^{(p_1,d_1)}$ and $g \in L^{(p_2,d_2)}$ where $\frac{1}{p_1} + \frac{1}{p_2} < 1$, then the product $h = fg$ belongs to $L^{(r,d_3)}$ where $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$, and $d_3 \geq 1$ satisfies $\frac{1}{d_1} + \frac{1}{d_2} \geq \frac{1}{d_3}$. Moreover,*

$$\|h\|_{(r,d_3)} \leq C(r)\|f\|_{(p_1,d_1)}\|g\|_{(p_2,d_2)}.$$

We need the following result.

Theorem 2.3 (Teorema 3.5, (O'NEIL, 1963)–Hölder inequality) *If P is a product operator*

$$h = P(f, g)$$

and if $f \in L^{(p,q_1)}$, $g \in L^{(p',q_2)}$, where

$$\frac{1}{p} + \frac{1}{p'} = 1 \text{ and } \frac{1}{q_1} + \frac{1}{q_2} \geq 1,$$

then $h \in L^1$ and,

$$\|h\|_1 \leq \|f\|_{(p,q_1)}\|g\|_{(p',q_2)}.$$

We have the following interpolation property between Lorentz spaces.

Proposition 2.4 (Theorem 5.3.1, (BERGH; LÖFSTRÖM, 1976)) *If $0 < p_0 < p_1 < \infty$, $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{P_0} + \frac{\theta}{P_1}$ and $1 \leq q_0, q_1, q \leq \infty$, then*

$$\left(L^{(p_0,q_0)}, L^{(p_1,q_1)}\right)_{\theta,q} = L^{(p,q)}.$$

A consequence is given by the following general Marcinkiewicz interpolation theorem.

Theorem 2.5 (Theorem 5.3.2, (BERGH; LÖFSTRÖM, 1976)) *Suppose that*

$$T : L^{(p_0,r_0)}(U, d\mu) \to L^{(q_0,s_0)}(V, d\nu), \tag{28}$$
$$T : L^{(p_1,r_1)}(U, d\mu) \to L^{(q_1,s_1)}(V, d\nu), \tag{29}$$

where $p_0 \neq p_1$ and $q_0 \neq q_1$. Put $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1/q = (1-\theta)/q_0 + \theta/q_1$. Then

$$T : L^{(p,r)}(U, d\mu) \to L^{(q,r)}(V, d\nu), \quad 0 < r \leqslant \infty.$$

In particular, we have

$$T : L^{p}(U, d\mu) \to L^{q}(V, d\nu).$$

We will need the following result.

Lemma 2.6 ((FERREIRA; VILLAMIZAR-ROA, 2006)) *Let be $1 \leq d \leq \infty$ and $1 < p \leq r < \infty$. There is a constant $C = C(r, n, \gamma, p) > 0$ such that, for all $\varphi \in L^{(p,d)}(R^n)$ and for all $t > 0$,*

$$\left\|\nabla_x^j G_\gamma(t)\varphi\right\|_{r,d} \leq Ct^{-\frac{n}{2\gamma}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{j}{2\gamma}}\|\varphi\|_{p,d},$$

where $j \in \{0\} \cup \mathbb{N}$.

3 Main results

In this section we will state and prove our main results. First, we give estimates for $G_{\alpha,\gamma}(t)$ and its derivatives in Lorentz spaces. Then, we obtain an Yamazaki-type estimate for $G_{\alpha,\gamma}(t)$.

Lemma 3.1 *Let be $1 < p \leq q < \infty$ and $0 < \frac{n}{2\gamma}\left(\frac{1}{p}-\frac{1}{r}\right) + \frac{1}{2\gamma} < 1$ then there is a constant $C > 0$ such that*

$$\sup_{t>0} t^{\frac{\alpha n}{2\gamma}\left(\frac{1}{p}-\frac{1}{r}\right)+\frac{\alpha}{2\gamma}} \left\|\frac{\partial}{\partial x_i} G_{\alpha,\gamma}(t)\phi\right\|_{(r,d)} \leq C\|\phi\|_{(p,d)}, \quad i = 1, 2, \cdots, n. \tag{30}$$

If $0 < \frac{n}{2\gamma}\left(\frac{1}{p}-\frac{1}{r}\right) < 1$, then there is a constant $C' > 0$ such that

$$\sup_{t>0} t^{\frac{\alpha n}{2\gamma}\left(\frac{1}{p}-\frac{1}{r}\right)} \left\|G_{\alpha,\gamma}(t)\phi\right\|_{(r,d)} \leq C'\|\phi\|_{(p,d)}. \tag{31}$$

Proof:

If $\phi \in L^{p,d}(\mathbb{R}^n)$, so for $1 < p \leq r < \infty$ and $1 \leq d \leq \infty$ it follows that

$$\begin{aligned}
\left\|\frac{\partial}{\partial x_i} G_{\alpha,\gamma}(t)\phi\right\|_{(r,d)} &\leq \int_0^\infty M_\alpha(\eta) \left\|\frac{\partial}{\partial x_i} G_\gamma(\eta t^\alpha)\phi\right\|_{(r,d)} d\eta && (32)\\
&\leq C\int_0^\infty M_\alpha(\eta)(\eta t^\alpha)^{-\frac{n}{2\gamma}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{2\gamma}}\|\phi\|_{(p,d)} d\eta && (33)\\
&= Ct^{-\frac{\alpha n}{2\gamma}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{\alpha}{2\gamma}} \int_0^\infty M_\alpha(\eta)\eta^{-\frac{n}{2\gamma}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{2\gamma}} d\eta \|\phi\|_{(p,d)}\\
&= Ct^{-\frac{\alpha n}{2\gamma}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{\alpha}{2\gamma}}\|\phi\|_{(p,d)}.
\end{aligned}$$

The inequality (32) follows from the definition of $G_{\alpha,\gamma}(t)$ and the derivation under the integral. In the inequality (33) we use the **Lemma 2.6** for $j = 1$. In the last equality the integral converges, as in equation (15), because the hypothesis $\frac{n}{2\gamma}\left(\frac{1}{p} - \frac{1}{r}\right) + \frac{1}{2\gamma} < 1$. The estimate (31) is obtained analogously using the **Lemma 2.6** for $j = 0$.

Theorem 3.2 (Yamazaki's Estimate) *Let* $1 < p < r < \infty$ *such that* $0 < \frac{n}{2\gamma}\left(\frac{1}{p} - \frac{1}{r}\right) + \frac{1}{2\gamma} < 1$. *There is* $C > 0$ *such that*

$$\int_0^\infty t^{\frac{n\alpha}{2\gamma}\left(\frac{1}{p}-\frac{1}{r}\right)+\frac{\alpha}{2\gamma}-1} \left\| \frac{\partial}{\partial x_i} G_{\alpha,\gamma}(t)\phi \right\|_{(r,1)} ds \leq C\|\phi\|_{(p,1)} \tag{34}$$

$\forall\, \phi \in L^{(p,1)}(\mathbb{R}^n)$.

Proof:

Define

$$\xi_{\alpha,\gamma}(t) = t^{\frac{n\alpha}{2\gamma}\left(\frac{1}{p}-\frac{1}{r}\right)+\frac{\alpha}{2\gamma}-1} \left\| \frac{\partial}{\partial x_i} G_{\alpha,\gamma}(t)\phi \right\|_{(r,1)} \tag{35}$$

and take $1 < p_1 < p < p_2 < r < \infty$ such that $\frac{1}{p} = \frac{\lambda}{p_1} + \frac{1-\lambda}{p_2}$, with $\lambda \in (0,1)$. So, from **Lemma 3.1**,

$$\left\| \frac{\partial}{\partial x_i} G_{\alpha,\gamma}(t)\phi \right\|_{(r,1)} \leq C t^{-\frac{n\alpha}{2\gamma}\left(\frac{1}{p_j}-\frac{1}{r}\right)-\frac{\alpha}{2\gamma}} \|\phi\|_{(p_j,1)}, \tag{36}$$

for each $\phi \in L^{(p_i,1)}$, $j \in \{1,2\}$. Define $l_j^{-1} = \frac{n\alpha}{2\gamma}\left(\frac{1}{p_j} - \frac{1}{p}\right) + 1$. So, from equation (35) and inequality (36), we have

$$\xi_{\alpha,\gamma}(t) \leq t^{-l_j^{-1}} \|\phi\|_{(p_j,1)}. \tag{37}$$

Since $\|t^{-l_j^{-1}}\|_{(l_j,\infty)} = 1$, from Example 2.1, it follows that

$$\|\xi_{\alpha,\gamma}(t)\|_{L^{(l_j,\infty)}(0,\infty)} \leq C\|\phi\|_{L^{(p_j,1)}(\mathbb{R}^n)}. \tag{38}$$

Note that $1 = \frac{\lambda}{l_1} + \frac{1-\lambda}{l_2}$. From the interpolation **Proposition 2.4**, follow that

$$L^{(p,1)}(\mathbb{R}^n) = (L^{(p_1,1)}(\mathbb{R}^n), L^{(p_2,1)}(\mathbb{R}^n))_{\lambda,1} \tag{39}$$

and

$$L^{(1,1)}(0,\infty) = (L^{(l_1,\infty)}(0,\infty), L^{(l_2,\infty)}(0,\infty))_{\lambda,1}. \tag{40}$$

So that we can use the general Marcinkiewicz interpolations **Theorem 2.5**, for $p_0 = p_1, q_0 = l_1, p_1 = p_2, q_1 = l_2, s_0 = s_1 = \infty\ r_0 = r_1 = r = 1$, $U = \mathbb{R}^n$ and $V = (0,\infty)$. Moreover, of inequality (38), we have

$$\xi_{\alpha,\gamma} : L^{(p_1,1)}(\mathbb{R}^n) \to L^{(l_1,\infty)}(0,\infty) \tag{41}$$

and

$$\xi_{\alpha,\gamma} : L^{(p_2,1)}(\mathbb{R}^n) \to L^{(l_2,\infty)}(0,\infty). \tag{42}$$

Remember that $\frac{1}{p} = \frac{\lambda}{p_1} + \frac{1-\lambda}{p_2}$ and $1 = \frac{\lambda}{l_1} + \frac{1-\lambda}{l_2}$, that is, $l = 1$. Thus, by means of the **Theorem 2.5** it follows

$$\xi_{\alpha,\gamma} : L^{(p,1)}(\mathbb{R}^n) \to L^{(1,1)}(0,\infty). \tag{43}$$

Therefore, we can write

$$\|\xi_{\alpha,\gamma}(t)\|_{L^1(0,\infty)} \leq C\|\phi\|_{(p,1)} \tag{44}$$

that is,

$$\int_0^\infty t^{\frac{n\alpha}{2\gamma}\left(\frac{1}{p}-\frac{1}{r}\right)+\frac{\alpha}{2\gamma}-1} \left\| \frac{\partial}{\partial x_i} G_{\alpha,\gamma}(t)\phi \right\|_{(r,1)} ds \leq \|\phi\|_{(p,1)}, \tag{45}$$

$\forall\ \phi \in L^{(p,1)}(\mathbb{R}^n)$.

4 Concluding remarks

With the fractional version of the Yamazaki estimate, **Theorem 3.2**, and the estimates given by **Lemma 3.1**, it is possible to study semilinear space-time fractional diffusion equation

$$u_t + D_t^{1-\alpha}(-\Delta)^\gamma u(t) = f(u), \qquad (x,t) \in \mathbb{R}^n \times (0,\infty), \tag{46}$$

where $\alpha, \gamma \in (0,1)$, and f is a nonlinear function with appropriate growth in L^p-weak spaces. We can refer, as an example, the generalization of the space fractional Burgers equation, given by equation

$$u_t = \nu(-\Delta)^{\beta/2}u - a \cdot \nabla(u^r), \tag{47}$$

where $r > 1$, $x \in \mathbb{R}^d$ $d = 1, 2, \cdots, t \geq 0$, $u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$, $\beta \in (0,2]$ and $a \in \mathbb{R}^d$ is a fixed vetor. Note that in **Lemma 3.1**, the following relation holds

$$\left\|G_{\alpha,\gamma}(t)\phi\right\|_{(r,d)} \leq t^{-\frac{\alpha n}{2\gamma}\left(\frac{1}{p}-\frac{1}{r}\right)} C'\|\phi\|_{(p,d)}. \tag{48}$$

So, as t increases, $G_{\alpha,\gamma}(t)$ decays in Lorentz space. Likewise we have the temporal decay for the partial derivatives of $G_{\alpha,\gamma}(t)$. Furthermore, it is noteworthy that we have the highest decay rate with the derivative.

Also note that it is $G_{\alpha,\gamma}(t)$, when $\alpha = 1$, it coincides with the operator $G_\gamma(t)$ as in (FERREIRA; VILLAMIZAR-ROA, 2006) and, when $\gamma = 1$, coincides with $G_\alpha(t)$, recovered in (CAICEDO et al., 2021).

Considering the physical applicability of the spatial Burgers equation (23), we continue the studies of the space time Burgers equation (22), so that the estimates developed in this work are important to prove the existence of mild solutions in L^p-weak spaces.

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