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Hadamard fractional integrals and derivatives with respect to functions

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Abstract

In this article, we study generalized fractional derivatives and integrals that contain kernels a generalized logarithm depending on a positive continuous function increasing. We generalize the Laplace transform in order to be applicable for the generalized fractional integrals and derivatives.

Keywords: Generalized fractional integrals, Generalized Caputo-Hadamard fractional derivative, Generalized Laplace transform.

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1 Introduction

Fractional calculus is a branch of classical mathematics, which deals with the generalization of operations of differentiation and integration to fractional order. There are numerous applications in a variety of fields such as viscoelasticity, porous media, modeling for physical phenomena exhibiting anomalous diffusion, and so on. There are different definitions of fractional derivatives available in the literature. However, the most commonly used is the Hadamard fractional derivative given by Hadamard (HADAMARD, 1892).

Motivated by diverse properties of Hadamard-type derivatives which are more generalized than the Hadamard fractional derivatives (ANATOLY, 2001; BUTZER; KILBAS; TRUJILLO, 2002a, 2002b, 2002c; OLIVEIRA, 2021) in this paper we consider a new fractional integrals called ϕ-Hadamard integral. These fractional integrals were motivated by Hadamard fractional integrals, this is, changing the kernels $(\ln t - \ln \tau)^{\alpha-1}/\Gamma(\alpha)$, by $(\ln_\phi(t) - \ln_\phi(\tau))^{\alpha-1}/\Gamma(\alpha)$, in the Hadamard fractional integrals, where $\ln_\phi$ is a generalized logarithm.

2 Preliminaries and Background Materials

In this section, we present some basic notations, definitions, and preliminary results, which will be used throughout this paper.

Let $[a, b]$ $(0 \leq a < b < \infty)$ be a finite interval on the half-axis $\mathbb{R}^+$ and $C[a, b]$ be the space of continuous functions on $[a, b]$. On the other hand, we have n-times absolutely continuous functions given by

$$AC^n[a, b] = \left\{ f : \Omega \to \mathbb{R};\ f^{(n-1)} \in AC\left([a, b]\right) \right\}. \tag{1}$$

Lemma 1 (MAGNUS, 2020) *Let $f\colon (a, b) \to \mathbb{R}$ be continuous and strictly increasing and*

$$m = \lim_{x \to 0^+} f(x) \text{ and } M = \lim_{x \to +\infty} f(x),$$

allowing $m = -\infty$ or $M = \infty$. Then the range of f is the interval (m, M) and f is a bijective function from $(0, +\infty)$ to (m, M). The inverse function $f^{-1}\colon (m, M) \to (a, b)$ is strictly increasing and continuous.

Definition 2.1 ($\mathfrak{D}$-function) *If a continuous function of $\phi : (0, \infty) \to \mathbb{R}$ is strictly positive and increasing, we call φ a $\mathfrak{D}$-class function, denoted by $\phi \in \mathfrak{D}$.*

The definition below is inspired by the work of J. Naudts (NAUDTS, 2004), and is an adapted version.

Definition 2.2 *Let $\phi \in \mathfrak{D}$. The ϕ-logarithm is defined as*

$$\ln_\phi(x) \quad = \quad \int_1^x \frac{1}{\phi(\tau)}\, \mathrm{d}\tau\ ,\ x > 0. \tag{2}$$

Remark 2 *Naudts' paper there is no requirement for continuity of function ϕ.*

It is important to be noted that if we set $\phi(x) = x$ in Definition (2.2), then ϕ-logarithm reduces to the natural logarithm $\log_e = \ln$. The b-base logarithm $\log_b(x)$ is obtained with $\phi(x) = x \ln(b)$. The $\phi-$logarithm shares several properties with the ordinary logarithm: it is strictly increasing, concave,

continuously differentiable, negative on $(0,1)$, positive on $(1,\infty)$, with $\ln_\phi(1)=0$. Of course, its values range between $m_\phi = -\int_0^1 \frac{1}{\phi(\tau)}\mathrm{d}\tau$ and $M_\phi = \int_1^{+\infty} \frac{1}{\phi(\tau)}\mathrm{d}\tau$. Also, the derivative of $\ln_\phi(x)$ is $1/\phi(x) > 0$, for all $x > 0$. Therefore, $\ln_\phi$ is continuous strictly increasing and so has an inverse function $\ln_\phi^{-1} = \exp_\phi : (m_\phi, M_\phi) \to (0,\infty)$. Moreover, $\exp_\phi$ is a continuous, strictly increasing and continuously differentiable function.

Lemma 3 *The function $y = \exp_\phi(x)$ is a solution to the differential equation*

$$\frac{\mathrm{d}}{\mathrm{d}x} y = \phi(y), \tag{3}$$

with the initial condition $y(0) = 1$.

Note that from some special choices of function ϕ we can find one-parameter logarithmic and exponential functions:

(i) For $\phi(x) = x^q$, $q \in \mathbb{R}^+ \setminus \{1\}$, we have the pair of Tsallis q-logarithm/q-exponential inverse functions of non-extensive statistical mechanics (see (TSALLIS, 1988; BORGES, 2004; OIKONOMOU; BAGCI, 2009; SINGH, 2016; TSALLIS, 2009; SILVA; MOREIRA; MORET, 2018; REIS; SILVA JUNIOR, 2021)):

$$\ln_q(x) = \int_1^x \frac{1}{\tau^q}\,\mathrm{d}\tau = \frac{x^{1-q}-1}{1-q}, \quad \exp_q(x) = [1+(1-q)x]^{\frac{1}{1-q}}. \tag{4}$$

Taking a limit $q \to 1$, these functions coincide with the standard logarithm and the standard exponential, respectively. In (EASTMOND, 2021), the author compare the Mittag-Leffler and q-exponential when $q \in (0,1)$ and $\alpha \in (0,1)$.

(ii) For $\phi(x) = \frac{2x}{x^\kappa + x^{-\kappa}}$, $\kappa \in (0,1)$, we get the κ-exponential and κ-logarithm functions proposed by Kaniadakis (KANIADAKIS, 2001, 2002; SILVA, 2021):

$$\ln_\kappa(x) = \int_1^x \frac{\tau^\kappa + \tau^{-\kappa}}{2}\frac{1}{\tau}\,\mathrm{d}\tau = \frac{1}{2\kappa}\left(x^\kappa - x^{-\kappa}\right), \quad \exp_\kappa(x) = \left(\kappa x + \sqrt{1+\kappa^2 x^2}\right)^{\frac{1}{\kappa}}, \tag{5}$$

which are limiting cases of both the ordinary exponential and logarithm functions as $\kappa \to 0$.

(iii) For $\phi(x) = \frac{q_{\mathrm{A}}^{-1} - q_{\mathrm{A}}}{(q_{\mathrm{A}}^{-1}-1)x^{q_{\mathrm{A}}^{-1}-2} - (q_{\mathrm{A}}-1)x^{q_{\mathrm{A}}-2}}$, we have Abe's one-parameter generalized logarithmic function; see (ABE, 1997; KANIADAKIS; LISSIA; SCARFONE, 2005):

$$\ln_{q_{\mathrm{A}}}(x) = \frac{x^{(q_{\mathrm{A}}^{-1})-1} - x^{q_{\mathrm{A}}-1}}{q_{\mathrm{A}}^{-1} - q_{\mathrm{A}}}, \tag{6}$$

which reduces to the standard logarithm for $q_{\mathrm{A}} \to 1$.

(iv) For $\phi(x) = (1+\epsilon x)x$ (see, (NAUDTS; ZHANG, 2018)):

$$\ln_\phi(x) = \ln\left(\frac{(1+\epsilon x)x}{1+\epsilon x}\right), \quad \exp_\phi(x) = \frac{1}{(1+\epsilon)e^{-x} - \epsilon}. \tag{7}$$

Moreover, other several special cases can be obtained as well. For a historical outline see (KANIADAKIS; LISSIA; SCARFONE, 2004; KANIADAKIS; LISSIA, 2004).

Remark 4 *A two-parameter generalization was introduced by Borges and Roditi (KANIADAKIS; LISSIA; SCARFONE, 2005; BORGES; RODITI, 1998) with*

$$\phi(x) = \frac{(k_1 - k_2)x}{k_1 x^{k_1} - k_2 x^{k_2}} \quad \text{and} \quad \ln^E x = \frac{x^{k_1} - x^{k_2}}{k_1 - k_2}. \tag{8}$$

Three interesting one-parameter logarithms cases can be retrieved:

(1) the Tsallis' logarithm $(k_1 = 1 - q,\ k_2 = 0)$: $\ln^E x = \ln_q x$;

(2) Kaniadakis' logarithm, $k_1 = \kappa$ *and* $k_2 = -\kappa$, $\ln^E x = \ln_\kappa x$.

(3) and Abe's logarithm $\ln^E x = \ln_{q_{\mathrm{A}}} x$, *where* $k_1 = 1/q_{\mathrm{A}} - 1$ *and* $k_2 = q_{\mathrm{A}} - 1$.

Definition 2.3 *Let* $\phi \in \mathfrak{D}^n$. *Then*

$$AC^n_\phi[a,b] = \left\{ f : [a,b] \to \mathbb{R} \text{ and } \delta^{n-1}_\phi f \in AC[a,b],\ \delta_\phi = \phi(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x} \right\}. \tag{9}$$

Conventionally, $AC^1_\phi[a,b] = AC[a,b]$.

We can also give a definition of limit form of δ_ϕ-derivative operator in the following way,

$$\left(\phi(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\right) f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{\ln \frac{\phi(x)+h}{\phi(x)}}. \tag{10}$$

Now we highlight some definitions of a fractional integrals and derivative of a function with respect to another function:

Definition 2.4 (SAMKO; KILBAS; MARICHEV, 1993; KILBAS; SRIVASTAVA; TRUJILLO, 2006) *Let* $\alpha > 0$, $n \in \mathbb{N}$, $f \in L_1[a,b]$ *and* $\psi \in C^n([a,b])$ *be an increasing function such that* $\psi'(x) \neq 0$, *for all* $x \in [a,b]$. *The* ψ*-Riemann–Liouville fractional integral and derivative of* f *with respect to* ψ *of order* α *are given by*

$$I^{\alpha;\psi}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (\psi(x) - \psi(\tau))^{\alpha-1} f(\tau)\, \psi'(\tau)\, \mathrm{d}\tau, \tag{11}$$

and

$$D^{\alpha;\psi}_{a+} f(x) = \left(\frac{1}{\psi'(\tau)} \cdot \frac{\mathrm{d}}{\mathrm{d}\tau}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^x (\psi(x) - \psi(\tau))^{n-\alpha-1} f(\tau)\, \psi'(\tau)\, \mathrm{d}\tau, \tag{12}$$

respectively, where $n = [\alpha] + 1$.

Definition 2.5 (ALMEIDA, 2017) *Let* $\alpha > 0$ *and* $f, \psi \in C^n([a,b]$ *with* ψ *an increasing function such that* $\psi'(x) \neq 0$, *for all* $x \in [a,b]$. *Then the* ψ*-Caputo fractional derivative of a function* f *with respect to* ψ *of order* α *is defined as*

$${}^C D^{\alpha;\psi}_{a+} f(x) = I^{n-\alpha;\psi}_{a+} \left(\frac{1}{\psi'(x)} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\right)^n f(x),$$

where $n = [\alpha] + 1$.

Property 5 (KILBAS; SRIVASTAVA; TRUJILLO, 2006; ALMEIDA, 2017) *Let* $f \in C^n[a,b]$, $\alpha > 0$ *and* $\lambda > 0$,

1. $f(x) = (\psi(x) - \psi(a))^{\lambda-1}$, *then*

$$I_{a+}^{\alpha;\psi} f(x) = \frac{\Gamma(\lambda)}{\Gamma(\alpha+\lambda)} (\psi(x) - \psi(a))^{\alpha+\lambda-1}.$$

2. $I_{a+}^{\alpha;\psi}\, I_{a+}^{\lambda;\psi} f(x) = I_{a+}^{\alpha+\lambda;\psi} f(x)$.

3. $I_{a+}^{\alpha;\psi}\, {}^{C}D_{a+}^{\alpha;\psi} f(x) = f(x) - \sum_{k=0}^{n-1} \left(\frac{1}{\psi'(x)} \cdot \frac{\mathrm{d}}{\mathrm{d}x} \right)^n f(a) \frac{(\psi(x) - \psi(a))^k}{k!}$.

3 Main results

Remark 6 *Note that the assumption that the function* ϕ *must be strictly positive on* $(0,\infty)$ *is sufficient to ensure that* $(\ln_\phi(x) - \ln_\phi(\tau)) > 0$ *when* $0 < \tau < x$.

We have the following characterization of the space $AC_\phi^n[a,b]$.

Theorem 7 *The space* $AC_\phi^n[a,b]$ *consists of those and only those functions* f *that are represented in the form*

$$f(x) = \frac{1}{(n-1)!} \int_a^x \big(\ln_\phi(x) - \ln_\phi(s)\big)^{n-1} \delta_\phi^n f(s) \frac{\mathrm{d}s}{\phi(s)} + \sum_{k=0}^{n-1} \frac{\delta_\phi^k f(a)}{k!} \big(\ln_\phi(x) - \ln_\phi(a)\big)^k, \tag{13}$$

where $\delta_\phi^n = \delta_\phi \cdot \delta_\phi^{n-1}$.

Proof Since $f \in AC_\phi^n[a,b]$, from Definition 2.3, $\delta_\phi^{n-1} f \in AC[a,b]$. Thus, there exists a function $g \in L_1[a,b]$, such that

$$\delta_\phi^{n-1} f(x) = \int_a^x g(s)\,\mathrm{d}s + c_{n-1}, \tag{14}$$

where c_{n-1} is a constant.

Dividing both sides of (14) by $\phi(x)$ and then integrating gives

$$\delta_\phi^{n-2} f(x) = \int_a^x \frac{1}{\phi(t)} \int_a^t g(u)\,\mathrm{d}u\,\mathrm{d}t + c_{n-1}\big(\ln_\phi(x) - \ln_\phi(a)\big) + c_{n-2}. \tag{15}$$

Changing the order of integration in the double integral on the right hand side of (15) and then integrating once gives

$$\delta_\phi^{n-2} f(x) = \int_a^x \big(\ln_\phi(x) - \ln_\phi(a)\big) g(s)\,\mathrm{d}s + c_{n-1}\big(\ln_\phi(x) - \ln_\phi(a)\big) + c_{n-2}. \tag{16}$$

Repeating the same procedure $n-2$ times, one gets

$$f(x) = \int_a^x \big(\ln_\phi(x) - \ln_\phi(a)\big)^{n-1} \frac{g(s)}{(n-1)!}\,\mathrm{d}s + \sum_{k=0}^{n-1} \frac{c_k}{k!} \big(\ln_\phi(x) - \ln_\phi(a)\big)^k. \tag{17}$$

It is clear from (14) that $g(s) = \phi(s)\delta_\phi^n f(s)$ and from the proof that

$$c_k = \delta_\phi^n f(a),\ k = 0, 1, \ldots, n-1.$$

This was the proof of the necessity. To proof the sufficiency it is enough to apply the operator δ_ϕ^{n-1} to both sides of (13). □

3.1 An integral operator involving the ϕ-logarithm function

In this section, we introduce a new fractional integral definition by generalizing the Hadamard fractional integral operator (HADAMARD, 1892). These operators were motivated by ψ-Riemann-Liouville fractional integrals with kernels given by ϕ-logarithms.

Definition 3.1 *Let $\alpha > 0$, f an integrable function defined on $[a, b]$ and $\phi \in \mathfrak{D}$. The ϕ-Hadamard fractional integrals of order α of f on $[a, b]$ is defined as*

$$\mathbb{I}_{a+}^{\alpha;\ln_\phi} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (\ln_\phi x - \ln_\phi \tau)^{\alpha-1} f(\tau) \frac{\mathrm{d}\tau}{\phi(\tau)}, \tag{18}$$

provided that the integral exists.

It is clear that (18) is a fractional integral of the form (11) with the function $\psi(x) = \ln_\phi(x)$, so that ϕ-deformed Hadamard fractional integration (18) is a "fractional integral of a function f with respect to the function ϕ".

Remark 8 *For $n \in \mathbb{N}$, $\mathbb{I}_{a+}^{n;\ln_\phi} f(x)$ can be expressed explicitly by the following n-fold integral*

$$\int_a^x \frac{\mathrm{d}\tau_1}{\phi(\tau_1)} \int_a^{\tau_1} \frac{\mathrm{d}\tau_2}{\phi(\tau_2)} \cdots \int_a^{\tau_{n-1}} f(\tau) \frac{\mathrm{d}\tau}{\phi(\tau)} = \frac{1}{(n-1)!} \int_a^x \left(\int_\tau^x \frac{1}{\phi(s)} \mathrm{d}s \right)^{n-1} f(\tau) \frac{\mathrm{d}\tau}{\phi(\tau)}. \tag{19}$$

Example 9 *For $\varphi(x) = x$, we recover the Hadamard fractional integral of order α:*

$$\mathbb{I}_{a+}^{\alpha;\ln} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (\ln x - \ln \tau)^{\alpha-1} f(\tau) \frac{\mathrm{d}\tau}{\tau}. \tag{20}$$

Example 10 *For $\varphi(x) = \frac{x}{x+1}$, we have*

$$\begin{aligned} \mathbb{I}_{a+}^{\alpha;{}^{\mathsf{N}}\ln} f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x \left({}^{\mathsf{N}}\ln x - {}^{\mathsf{N}}\ln \tau \right)^{\alpha-1} (\tau+1) f(\tau) \frac{\mathrm{d}\tau}{\tau} \\ &= \frac{1}{\Gamma(\alpha)} \int_a^x (\ln x - \ln \tau + x - \tau)^{\alpha-1} (\tau+1) f(\tau) \frac{\mathrm{d}\tau}{\tau}, \end{aligned} \tag{21}$$

where ${}^{\mathsf{N}}\ln x = x - 1 + \ln x$ is the function proposed by Nigel J. Newton (NEWTON, 2012).

Motivated by several authors who have introduced one-parameter families of logarithms in the context of generalized statistical mechanics. In the following we obtain four new fractional integral formulas stated in below.

Example 11 *For $\varphi(x) = x^q$, $q \in \mathbb{R} \setminus \{1\}$, we obtain the so-called Katugampola fractional integral; see (KATUGAMPOLA, 2011):*

$$\begin{aligned} \mathbb{I}_{a+}^{\alpha;\ln_q} f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln_q x - \ln_q \tau \right)^{\alpha-1} f(\tau) \frac{\mathrm{d}\tau}{\tau^q} \\ &= \frac{1}{\Gamma(\alpha)} \int_a^x \left(\frac{x^{1-q} - \tau^{1-q}}{1-q} \right)^{\alpha-1} f(\tau) \frac{\mathrm{d}\tau}{\tau^q}. \end{aligned} \tag{22}$$

Example 12 *For* $\phi(x) = \frac{2x}{x^{\kappa}+x^{-\kappa}}$ *and* $\kappa \in (0,1)$, *we have*

$$\begin{aligned} \mathbb{I}_{a+}^{\alpha;\ln_\kappa} f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (\ln_\kappa x - \ln_\kappa \tau)^{\alpha-1} (\tau^\kappa + \tau^{-\kappa}) f(\tau) \frac{\mathrm{d}\tau}{2\tau} \\ &= \frac{1}{\Gamma(\alpha)} \int_a^x (x^\kappa - \tau^\kappa - x^{-\kappa} + \tau^{-\kappa})^{\alpha-1} \frac{\tau^{\kappa-1} + \tau^{-\kappa-1}}{4\kappa} f(\tau)\, \mathrm{d}\tau. \end{aligned} \tag{23}$$

Example 13 *For* $\phi(x) = \frac{q_{\mathrm{A}}^{-1} - q_{\mathrm{A}}}{(q_{\mathrm{A}}^{-1}-1)x^{q_{\mathrm{A}}^{-1}-2} - (q_{\mathrm{A}}-1)x^{q_{\mathrm{A}}-2}}$, $q_{\mathrm{A}} \neq 1$, *we have*

$$\mathbb{I}_{a+}^{\alpha;\ln^E} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln^E x - \ln^E \tau\right)^{\alpha-1} \frac{(q_{\mathrm{A}}^{-1}-1)\tau^{q_{\mathrm{A}}^{-1}-2} - (q_{\mathrm{A}}-1)\tau^{q_{\mathrm{A}}-2}}{q_{\mathrm{A}}^{-1} - q_{\mathrm{A}}} f(\tau)\, \mathrm{d}\tau. \tag{24}$$

Example 14 *For* $\phi(x) = (1+\epsilon x)x$ *and* $\epsilon \geq 0$,

$$\mathbb{I}_{a+}^{\alpha;\ln_\phi} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{(1+\epsilon x)x}{1+\epsilon x} - \ln \frac{(1+\epsilon\tau)\tau}{1+\epsilon\tau}\right)^{\alpha-1} \frac{f(\tau)}{(1+\epsilon\tau)\tau}\, \mathrm{d}\tau. \tag{25}$$

It is noted that if we set $q \to 1$, $\kappa \to 0$, $q_{\mathrm{A}} \to 1$ and $\epsilon = 0$ yields the original Hadamard fractional integral.

Theorem 15 *Let* $\alpha > 0$ *and* $\beta > 0$. *Then, we have the following semigroup property given by*

$$\mathbb{I}_{a+}^{\alpha;\ln_\phi} \mathbb{I}_{a+}^{\beta;\ln_\phi} f(x) = \mathbb{I}_{a+}^{\alpha+\beta;\ln_\phi} f(x), \ \forall x \in (a,b).$$

Proof See (KILBAS; SRIVASTAVA; TRUJILLO, 2006). □

4 Caputo-type ϕ-Hadamard fractional derivatives

Analogous to the Hadamard fractional calculus, the ϕ-Caputo-Hadamard fractional derivative is defined in the following way.

Definition 4.1 *Let* $\alpha > 0$, $n \in \mathbb{N}$, $f \in C^n[a,b]$ *and* $\phi \in \mathfrak{D} \cap C^{n-1}(0,\infty)$. *The* ϕ*-Caputo-Hadamard fractional derivatives of* f *of order* α *are given by*

$${}^{\mathrm{CH}}\mathbb{D}_{a+}^{\alpha;\ln_\phi} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (\ln_\phi t - \ln_\phi \tau)^{n-\alpha-1} \left(\phi(\tau) \cdot \frac{\mathrm{d}}{\mathrm{d}\tau}\right)^n f(\tau) \frac{\mathrm{d}\tau}{\phi(\tau)}, \tag{26}$$

with $n-1 \leq \alpha < n$.

Theorem 16 *The following relations hold:*

$$\lim_{\alpha \to n^-} {}^{\mathrm{CH}}\mathbb{D}_{a+}^{\alpha;\ln_\phi} f(t) = \delta_\phi^n f(t) \ \textit{and} \ \lim_{\alpha \to (n-1)-} {}^{\mathrm{CH}}\mathbb{D}_{a+}^{\alpha;\ln_\phi} f(t) = \delta_\phi^{n-1} f(t) - \delta_\phi^{n-1} f(a). \tag{27}$$

Proof Using the Definition (4.1) and applying integrating by parts formula $u = \delta_\phi f(t)$ and $\mathrm{d}v = [(\ln_\phi(t) - \ln_\phi(\tau))^{n-\alpha}/\phi(\tau)]\,\mathrm{d}\tau$, one gets

$$
\begin{aligned}
&{}^{\mathrm{CH}}\mathbb{D}_{a+}^{\alpha;\ln_\phi} f(t) \\
&= \frac{1}{\Gamma(n+1-\alpha)}\left\{(\ln_\phi(t) - \ln_\phi(a))^{n-\alpha}\delta_\phi^n f(a) + \int_a^x (\ln_\phi(t) - \ln_\phi(a))^{n-\alpha}\mathrm{d}[\delta_\phi^n f(\tau)]\right\}. \quad (28)
\end{aligned}
$$

Thus,

$$
\lim_{\alpha\to n^-} {}^{\mathrm{CH}}\mathbb{D}_{a+}^{\alpha;\ln_\phi} f(t) = \delta_\phi f(a) + [\delta_\phi^n f(t) - \delta_\phi^n f(a)] = \delta_\phi^n f(t). \quad (29)
$$

For the second formula, starting with the definition, we obtain directly that

$$
\lim_{\alpha\to(n-1)+} {}^{\mathrm{CH}}\mathbb{D}_{a+}^{\alpha;\ln_\phi} f(t) = \int_a^t \frac{\mathrm{d}}{\mathrm{d}\tau}\delta_\phi^{n-1} f(\tau)\,\mathrm{d}\tau = \delta_\phi^{n-1} f(t) - \delta_\phi^{n-1} f(a). \quad (30)
$$

The other two formulas are proven in a similar way. □

5 The ϕ-Laplace transform

Definition 5.1 *Let $f : [a,\infty) \to \mathbb{R}$ be a real-valued function and $\phi \in \mathfrak{D}$. Then the Laplace transform of f with respect to ϕ is defined by*

$$
\mathcal{L}_\phi\{f(t)\}(s) = F_\phi(s) = \int_a^\infty e^{-s\left(\ln_\phi \tau - \ln_\phi a\right)} f(\tau)\,\frac{\mathrm{d}\tau}{\phi(\tau)}, \quad (31)
$$

for all values of $s \in \mathbb{C}$ such that this integral converges.

Theorem 17 *Let f_1 and f_2 be two functions whose ϕ-Laplace transform exist for $s > c_1$ and $s > c_2$ respectively. Then for s greater than the maximum of c_1 and c_2,*

$$
\mathcal{L}_\phi\{a_1 f_1(t) + a_2 f_2(t)\}(s) = a_1\mathcal{L}_\phi\{f_1(t)\}(s) + a_2\mathcal{L}_\phi\{f_2(t)\}(s) \quad (32)
$$

where a_1, a_2 are real constants.

Proof Direct consequence of Definition (5.1). □

Theorem 18 *Let $\phi \in \mathfrak{D}$ and $f : [0,\infty) \to \mathbb{R}$ be a function such that $\mathcal{L}_\phi\{f(t)\}(s) = F_\phi(s)$ exists. Then:*

$$
F_\phi(s) = \begin{cases} \int_0^{M_\phi - \ln_\phi a} e^{-s\tau} f(\tau)\,\mathrm{d}\tau & \text{for } M_\phi < \infty \\ \mathcal{L}\left\{f\left(\exp_\phi(t + \ln_\phi a\right)\right\}(s) & \text{for } M_\phi = \infty, \end{cases} \quad (33)
$$

where $\mathcal{L}[f(t)](s) = \int_0^\infty e^{-s\tau} f(\tau)\,\mathrm{d}\tau$ and $M_\phi = \int_1^\infty \frac{1}{\phi(\tau)}\,\mathrm{d}\tau$.

Proof The proof is straight forward if one use the change of variable $u = \ln_\phi t - \ln_\phi a$ in (31). □

Definition 5.2 *In (MITTAG-LEFFLER, 1903), Gösta Mittag-Leffler introduced the well-known Mittag-Leffler function $E_\alpha(z)$, given by*

$$E_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)}, \quad \alpha \in \mathbb{C}, \quad Re(\alpha) > 0. \tag{34}$$

Later on, a natural generalisation of $E_\alpha(z)$ was discussed by Wiman in (WIMAN, 1905). He introduced the function $E_{\alpha,\nu}(z)$ as

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, \quad \alpha, \beta \in \mathbb{C}, \quad Re(\alpha) > 0. \tag{35}$$

If we consider $\beta = 1$ in (35), we obtain the Mittag-Leffler function (34).

Lemma 19 *Let $\ln_\phi$ be a unbounded function. Then, we have the following results.*

1. $\mathcal{L}_\phi \{1\}(s) = \frac{1}{s}$, $s > 0$.
2. $\mathcal{L}_\phi \left\{\left(\ln_\phi t - \ln_\phi a\right)^\alpha\right\}(s) = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$, $\alpha > -1$, $s > 0$.
3. $\mathcal{L}_\phi \left\{e^{\lambda \ln_\phi t}\right\}(s) = \frac{e^{\lambda \ln_\phi t}}{s-\lambda}$, $s > \lambda$.
4. $\mathcal{L}_\phi \left\{E_\alpha\left(\lambda\left(\ln_\phi t - \ln_\phi a\right)^\alpha\right)\right\}(s) = \frac{s^{\alpha-1}}{s^\alpha - \lambda}$, $\left|\frac{\lambda}{s^\alpha}\right| < 1$, *and*
5. $\mathcal{L}_\phi \left\{\left(\ln_\phi t - \ln_\phi a\right)^{\alpha k+\beta-1} E_{\alpha,\beta}^{(k)}\left(\pm\xi\left(\ln_\phi t - \ln_\phi a\right)^\alpha\right)\right\}(s) = \frac{k! s^{\alpha-\beta}}{(s^\alpha \mp \xi)^{k+1}}$, $\Re(s) > |\xi|^{\frac{1}{\alpha}}$, *see (PODLUBNY, 1999, eq. 1.80).*

Definition 5.3 *A function $f : [0, \infty) \to \mathbb{R}$ is said to be of $\ln_\phi$-exponential order if there exist non-negative constants M, c, T such that $|f(t)| \leq M e^{c \ln_\phi t}$ for $t \geq T$.*

Now we can state the existence theorem for ϕ-Laplace transform.

Theorem 20 *If $f : [a, \infty) \to \mathbb{R}$ is a piecewise continuous function and is of $\ln_\phi$-exponential order, then the ϕ-Laplace transform of f always exists for $s > c$.*

Proof For the $\ln_\phi$-exponentially bounded function f, we have

$$\begin{aligned}
\left|F_\phi(s)\right| &= \left|\int_a^\infty e^{-s\left(\ln_\phi \tau - \ln_\phi a\right)} f(\tau) \frac{\mathrm{d}\tau}{\phi(\tau)}\right| \\
&\leq \int_a^\infty e^{-s\left(\ln_\phi \tau - \ln_\phi a\right)} |f(\tau)| \frac{\mathrm{d}\tau}{\phi(\tau)} \\
&\leq M \int_a^\infty e^{-s\left(\ln_\phi \tau - \ln_\phi a\right)} e^{c \ln_\phi \tau} \frac{\mathrm{d}\tau}{\phi(\tau)} \\
&= M e^{c \ln_\phi a} \left[\frac{e^{-(s-c)\ln_\phi \tau}}{-(s-c)}\right]_a^\infty < +\infty,
\end{aligned}$$

where in the last line we used the assumption that $s > c$. This completes the proof. □

Remark 21 *From the above proof, it follows that* $\lim_{s\to\infty} F_\phi(s) = 0$ *for any* $\ln_\phi$*-exponentially bounded function* f.

Definition 5.4 *The inverse* ϕ*-Laplace transform of* $F_\phi(s)$ *is given by*

$$f(t) = \mathcal{L}_\phi^{-1}\left\{F_\phi(s)\right\}(t) = \int_{c-i\infty}^{c+i\infty} e^{s\left(\ln_\phi t - \ln_\phi a\right)} F_\phi(s)\mathrm{d}s,\ \ c > 0,\ \ i^2 = -1. \tag{36}$$

The following theorem gives the ϕ-Laplace transform of δ_ϕ derivative.

Theorem 22 *Let the function* $f \in AC_\phi[a,T]$ *of* $\ln_\phi$*-exponential order such that* $\delta_\phi f$ *is piecewise continuous over every finite interval* $[a,T]$*, and* $\ln_\phi$ *not bounded. Then* ϕ*-Laplace transform of* $\delta_\phi f$ *exists,*

$$\mathcal{L}_\phi\left\{\delta_\phi f(t)\right\}(s) = s\mathcal{L}_\phi\left\{f(t)\right\}(s) - f(a). \tag{37}$$

Proof One can easily see the proof by using the definition of ϕ-Laplace transform the change of variable $u = \ln_\phi t - \ln_\phi a$. □

Theorem (22) can be generalized as follows.

Theorem 23 *Let* $f \in AC_\phi^{n-1}[a,T]$ *such that* $\delta_\phi^i f$, $i = 0,1,2,\ldots,n-1$ *are* $\ln_\phi$*-exponential order. Let* $\delta_\phi^n f$ *be a piecewise continuous function on the interval* $[a,T]$*, and* $\ln_\phi$ *not bounded. Then, the* ϕ*-Laplace transform of* $\delta_\phi^n f$ *exists and*

$$\mathcal{L}_\phi\left\{\delta_\phi^n f(t)\right\}(s) = s^n F_\phi(s) - \sum_{k=0}^{n-1} s^{n-k-1}\delta_\phi^k f(a). \tag{38}$$

Proof The proof can be done by mathematical induction. □

Based on (JARAD; ABDELJAWAD, 2020), we introduce the following definition.

Definition 5.5 *Let* f *and* g *be of* $\ln_\phi$*-exponential order, piecewise continuous functions over each finite interval* $[a,T]$*. Then, the* ϕ*-convolution of* f *and* g *is the function* $f *_\phi g$ *defined by*

$$\left(f *_\phi g\right)(t) = \int_a^t f\left(\exp_\phi(\ln_\phi t + \ln_\phi a - \ln_\phi \tau)\right) g(\tau)\,\frac{\mathrm{d}\tau}{\phi(\tau)}. \tag{39}$$

Lemma 24 *Let* f *and* h *be two functions which are piecewise continuous at each interval* $[a,T]$ *and of* $\ln_\phi$*-exponential order. Then*

$$\left(f *_\phi g\right)(t) = \left(g *_\phi f\right)(t). \tag{40}$$

Theorem 25 (Convolution property) *Let* $\ln_\phi$ *be a unbounded function,* f *and* g *be two functions which are piecewise continuous at each interval* $[a,T]$ *and of* $\ln_\phi$*-exponential order. Then*

$$\mathcal{L}_\phi\left\{\left(f *_\phi g\right)(t)\right\} = \mathcal{L}_\phi\left\{f(t)\right\}(s)\,\mathcal{L}_\phi\left\{g(t)\right\}(s). \tag{41}$$

Theorem 26 *Let* $\ln_\phi$ *be a unbounded function,* $\alpha > 0$ *and* f *be a piecewise continuous function on each interval* $[a,t]$ *and of* $\ln_\phi$*-exponential order. Then*

$$\mathcal{L}_\phi\left\{\mathbb{I}_{a+}^{\alpha;\ln_\phi} f(t)\right\}(s) = \frac{\mathcal{L}_\phi\left\{f(t)\right\}(s)}{s^\alpha}. \tag{42}$$

Proof By means of Definition (5.5) for convolution, ϕ-Hadamard integral can be rewritten as

$$\mathbb{I}_{a+}^{\alpha;\ln_\phi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (\ln_\phi t - \ln_\phi \tau)^{\alpha-1} f(\tau) \frac{\mathrm{d}\tau}{\phi(\tau)} = \frac{1}{\Gamma(\alpha)} \left(\ln_\phi t - \ln_\phi a \right)^{\alpha-1} *_\phi f(t). \tag{43}$$

Thus from convolution property given in Theorem (25), Theorem (17) and Lemma (17), we have

$$\begin{aligned} \mathcal{L}_\phi \left\{ \mathbb{I}_{a+}^{\alpha;\ln_\phi} f(t) \right\}(s) &= \frac{1}{\Gamma(\alpha)} \mathcal{L}_\phi \left\{ \left(\ln_\phi t - \ln_\phi a \right)^{\alpha-1} *_\phi f(t) \right\}(s) \\ &= \frac{1}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{s^\alpha} \mathcal{L}_\phi \{ f(t) \}(s) \\ &= \frac{\mathcal{L}_\phi \{ f(t) \}(s)}{s^\alpha}. \end{aligned} \tag{44}$$

□

The following theorem states the ϕ-Laplace transform of the ϕ-Caputo-Hadamard fractional derivative.

Theorem 27 *Let a bounded function $\phi \in \mathfrak{D}$, $\alpha > 0$ and $f \in AC_\phi^n[a,b]$ and $\delta_\phi^k f$, $k = 0, 1, \ldots, n$ be of $\ln_\phi$-exponential order. Then*

$$\mathcal{L}_\phi \left\{ {}^{\mathrm{CH}}\mathbb{D}_{a+}^{\alpha;\ln_\phi} f(t) \right\}(s) = s^\alpha \left[F_\phi(s) - \sum_{k=0}^{n-1} s^{-k-1} \delta_\phi^k f(a^+) \right]. \tag{45}$$

Proof For the $\ln_\phi$-exponentially bounded function f, we have

$$\begin{aligned} \mathcal{L}_\phi \left\{ {}^{\mathrm{CH}}\mathbb{D}_{a+}^{\alpha;\ln_\phi} f(t) \right\}(s) &= \mathcal{L}_\phi \left\{ \mathbb{I}_{a+}^{n-\alpha;\ln_\phi} \delta_\phi^n f(t) \right\}(s) \\ &= \frac{\mathcal{L}_\phi \left\{ \delta_\phi^n f(t) \right\}(s)}{s^{n-\alpha}} \\ &= s^{\alpha-n} \left[s^n F_\phi(s) - \sum_{k=0}^{n-1} s^{n-k-1} \delta_\phi^k f(a) \right] \\ &= s^\alpha \left[F_\phi(s) - \sum_{k=0}^{n-1} s^{-k-1} \delta_\phi^k f(a^+) \right]. \end{aligned} \tag{46}$$

□

6 Applications

6.1 ϕ-Caputo-Hadamard linear fractional differential equations

Theorem 28 *For $0 < \alpha \leq 1$, $\ln_\phi$ not bounded and $\lambda, b \in \mathbb{R}$, the fractional initial value problem*

$$\begin{cases} {}^{\mathrm{CH}}\mathbb{D}_{a+}^{\alpha;\ln_\phi} y(t) - \lambda y(t) &= f(t), \\ y(a) &= b, \end{cases} \tag{47}$$

has the solution

$$y(t) = bE_\alpha \left(\lambda \left(\ln_\phi t - \ln_\phi a \right)^\alpha \right) + \int_a^t \left(\ln_\phi t - \ln_\phi \tau \right)^{\alpha-1} E_{\alpha,\alpha} \left(\lambda \left(\ln_\phi t - \ln_\phi \tau \right)^\alpha \right) f(\tau) \frac{\mathrm{d}\tau}{\phi(\tau)}. \tag{48}$$

Proof The proof of this theorem follows directly from the definition of the Laplace transform to both sides of the equation (47) and then using Theorem (27) with $n = 1$, convolution property, and Lemma (19).

Remark 29 *Similar results can be proved for fractional-order differential equations in the settings of Caputo fractional derivatives (KEXUE; JIGEN, 2011). Moreover, for the case $\phi(t) = t$, the solution 48 coincides with the solution given in equation (26) in (MA, 2019).*

□

6.2 Transformation between ϕ-Caputo–Hadamard and Caputo Fractional Derivatives

Now, we present a transformation reducing the ϕ-Caputo–Hadamard fractional problems to their ϕ-Caputo analogues (see related works (GOHAR; LI; LI, 2020) and (ZHENG, 2021)). It follows from the definitions of ϕ-Caputo–Hadamard and Caputo derivatives with $0 < \alpha < 1$ that:

$$\begin{aligned}
{}^{\mathrm{CH}}\mathbb{D}_{a+}^{\alpha;\ln_\phi} y(t) &= \frac{1}{\Gamma(1-\alpha)} \int_a^t (\ln_\phi t - \ln_\phi \tau)^{-\alpha} \delta_\phi y(\tau) \frac{\mathrm{d}\tau}{\phi(\tau)} \\
&= \frac{1}{\Gamma(1-\alpha)} \int_a^t (\ln_\phi t - \ln_\phi a - \ln_\phi \tau + \ln_\phi a)^{-\alpha} \delta_\phi y(\tau)\, \mathrm{d}\left(\ln_\phi \tau - \ln_\phi a\right) \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^{\ln_\phi t - \ln_\phi a} (\ln_\phi t - \ln_\phi a - s)^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}s} y\left(\exp_\phi\left(s + \ln_\phi a\right)\right) \mathrm{d}s \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^u (u-s)^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}s} y\left(\exp_\phi\left(s + \ln_\phi a\right)\right) \mathrm{d}s \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^u (u-s)^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}s} \tilde{y}_{a;\phi}(s)\, \mathrm{d}s \quad \left(y\left(\exp_\phi\left(s + \ln_\phi a\right)\right) = \tilde{y}_{a;\phi}(s)\right) \\
&= {}^{\mathrm{C}}\mathbb{D}_{0+}^{\alpha} \tilde{y}_{a;\phi}(u).
\end{aligned} \tag{49}$$

Theorem 30 *Suppose that $f(t, y)$ is continuous. Then the Cauchy problem*

$$\begin{cases} {}^{\mathrm{CH}}\mathbb{D}_{1+}^{\alpha;\ln_\phi} y(t) = f(t, y), \quad 1 < t, \ 0 < \alpha < 1, \\ y(1) = y_1, \ y_1 \in \mathbb{R}, \end{cases} \tag{50}$$

is equivalent to the following Volterra integral equation

$$y(t) = y_1 + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln_\phi t - \ln_\phi s\right)^{\alpha-1} f(s, y(s)) \frac{\mathrm{d}s}{\phi(s)}. \tag{51}$$

Proof From the relation (49), it is evident that the initial value problem with Caputo derivative

$$\begin{cases} {}^{\mathrm{C}}\mathbb{D}_{0+}^{\alpha} \tilde{y}_{1;\phi}(u) = f(\exp_\phi u, \tilde{y}_{1;\phi}), \quad u > 0, \\ \tilde{y}_{1,\phi}(0) = y_1, \end{cases} \tag{52}$$

is equivalent to the initial value problem with ϕ-Caputo–Hadamard derivative

$$\begin{cases} {}^{\mathrm{CH}}\mathbb{D}_{1+}^{\alpha;\ln_\phi} y(t) = f(t, y), \quad t > 1, \\ y(1) = y_1, \end{cases} \tag{53}$$

where $0 < \alpha < 1$, $f(t, y)$ is a nonlinear function, and initial value y_1 is given. It immediately follows from (52) that:

$$\tilde{y}_{1;\phi}(u) = y_1 + \frac{1}{\Gamma(\alpha)} \int_0^u (u - \tau)^{\alpha-1} f(\exp_\phi(u), \tilde{y}_{1;\phi}(\tau))\, \mathrm{d}\tau \tag{54}$$

Therefore:

$$y(t) = y(\exp_\phi(\ln_\phi x)) = \tilde{y}_{1;\phi}(u) \tag{55}$$

solves (49). □

Remark 31 *In particular for Equation 51,*

- *If $\phi(t) \longrightarrow t$, then it corresponds to Hadamard fractional Volterra integral equation (GOHAR; LI; YIN, 2020; ADJABI et al., 2016).*
- *If $\phi(t) \longrightarrow t^{1-\rho}$ and $\rho < 1$, then corresponds to Caputo-Katugampola fractional Volterra integral equation, as shown by Odibat e Baleanu (ODIBAT; BALEANU, 2020).*

7 Conclusions

The main objective of this article was to introduce a new fractional integral that generalizes the generalized conformable, Hadamard and Katugampola fractional integrals, into a single form with generalized logarithm kernel. The ϕ-logarithm functions depend on the choice of a continuous positive increasing function ϕ defined on $(0, \infty)$, but has many properties of the natural logarithm function. Was studied generalization of the Laplace transform and also convolution. We have shown that the solution representations are expressible by means of the Mittag-Leffler functions and coincide with the results found in the literature for special cases of ϕ. In addition, some fractional differential equations are simply solved analytically by a given definition which exactly agrees with the classical fractional of the Caputo, Caputo-Hadamard and Katugampola-Caputo.

We can introduce different classes of differential and integral operators with ϕ-logarithmic kernel acting on a sufficiently well-behaved function f as

1. ϕ-Hadamard-type integral and differential:

$$^{\mu}\mathbb{I}_{a+}^{\alpha;\ln_\phi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{\tau}{t}\right)^{\mu} (\ln_\phi t - \ln_\phi \tau)^{\alpha-1} f(\tau) \frac{\mathrm{d}\tau}{\phi(\tau)}. \tag{56}$$

2. ϕ-Hadamard proportional integral and differential:

$$\mathbb{H}_{a+}^{\alpha,\beta;\ln_\phi} f(t) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_a^t e^{[\frac{\beta-1}{\beta}(\ln_\phi t - \ln_\phi \tau)]} (\ln_\phi(t) - \ln_\phi(\tau))^{\alpha-1} f(\tau) \frac{\mathrm{d}\tau}{\phi(\tau)}. \tag{57}$$

3. ϕ-Integro-differential operator:

$$\widehat{\mathbb{O}}_{\alpha}^{\ln_\phi} f(t) = \frac{1}{\Gamma(\alpha)} \int_{\frac{1-a}{b}}^t \left(\ln_\phi(a + bt) - \ln_\phi(a + b\tau)\right)^{\alpha-1} f(\tau) \frac{b}{\phi(a + b\tau)}\, \mathrm{d}\tau. \tag{58}$$

4. ϕ-Hadamard integral and differential:

$$\widehat{\mathbb{I}}_{\alpha}^{\ln_\phi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln^E t - \ln^E \tau\right)^{\alpha-1} f(\tau) \frac{\tau^{k_1-1} - \tau^{k_2-1}}{k_1 - k_2}\, \mathrm{d}\tau. \tag{59}$$

5. ϕ-Hilfer differential:

$$\mathbb{D}_{a+}^{\alpha,\beta;\ln_\phi} f(t) = \mathbb{I}_{a+}^{\beta(n-\alpha);\ln_\phi} \left(\phi(t) \cdot \frac{\mathrm{d}}{\mathrm{d}t}\right)^n \mathbb{I}_{a+}^{(1-\beta)(n-\alpha);\ln_\phi} f(t). \tag{60}$$

As a result we get a more generalized fractional operator, i.e. the generalization of all the Katugampola operators, generalized conformable fractional operators, Hadamard fractional operators and so on, all with an ϕ-logarithm kernel.

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