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Discret fractional SIR model applied in dynamic of COVID-19

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Abstract

In this article we present the discrete SIR fractional model, as an alternative to the study of epidemiologic problems and developing mathematical modeling. In particular, we applied the fractional model to determine the basic reproduction number R_0 of COVID-19 by adjusting curves over a period of the 250 days. The time considered is relative to the first wave of COVID-19 in 2020 in the state of Maranhão, Brazil. With the values of R_0 it is possible to know better this disease so that more efficient measures are taken against it.

Keywords: COVID-19. curve fitting. discret Caputo derivative. fractional SIR model. basic reproduction number R_0.

1 Introduction

The pandemic is the propagation of disease in many individuals (REZENDE, 1998). It considers phenomena that modify one or more characteristics of a group. In general, virus, bacteria, or other microorganisms are responsible for alterations in the health of a population, turning possible the pandemic. Therefore, it is fundamental to investigate and control the problem. In this context, mathematical epidemiology is the area that studies the spread of that diseases, based on mathematical models and computational methods, that possibilities a better comprehension of disease.

The work presents the fractional SIR (Susceptible-Infected-Removed) model as an alternative to getting essential reporting in the description of the dissemination of coronavirus disease (COVID-19), which is an infectious disease caused by the SARS-CoV-2 virus. The study is one extension of the paper (COSTA et al., 2021) that investigates the dynamic of the COVID-19 in the state of Maranhão in Brazil, considering the classical model. The fractional model shows variations of the classic model SIR, based on discretization of Caputo fractional derivative (LI; ZENG, 2015; PŁOCINICZAK, 2022). It permits the investigation and obtaining of the basic reproduction number R_0 and infection rate, which are fundamental to combat a pandemic. The structure of the article is: section 1 presents preliminaries results, that are Caputo derivative and discrete form; section 2 shows the classical SIR model and its properties; section 3 gives the fractional SIR model; section 4 presents applications of the fractional in the COVID-19; Concluding and remarking are presents to finish the work.

2 Fractional operators

Definition 1 (Riemann-Liouville integral (OLIVEIRA, 2019)) *The Riemann–Liouville fractional integral of $y(t)$ with order $\alpha > 0$ is given by the expression:*

$$\mathcal{I}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds \tag{1}$$

Definition 2 (Caputo derivative (OLIVEIRA, 2019)) *The Caputo fractional derivative of $y(t)$ with order $0 < \alpha < 1$ is given by:*

$${}^{C}\mathcal{D}^{\alpha} y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} y'(s) ds \tag{2}$$

The discretization of the fractional operators are given by(LI; ZENG, 2015; PŁOCINICZAK, 2022):

$$\mathcal{I}^{\alpha} y_n = \widehat{\mathcal{I}}^{\alpha} y_n + Q_n \tag{3}$$

$${}^{C}\mathcal{D}^{\alpha} y_n = {}^{C}\widehat{\mathcal{D}}^{\alpha} y_n + R_n, \tag{4}$$

in which $y_n := y(t_n)$, $t_n = nh$, $h > 0$ and the discretizations $\widehat{\mathcal{I}}^{\alpha}$ and $\widehat{\mathcal{D}}^{\alpha}$ are defined by

$$\widehat{\mathcal{I}}^{\alpha} y_n = \frac{h^{\alpha}}{\Gamma(1+\alpha)} \sum_{i=0}^{n-1} b_{n-i}(\alpha) y_{i+1} \tag{5}$$

$$\widehat{\mathcal{D}}^{\alpha} y_n = \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{i=0}^{n-1} b_{n-i}(1-\alpha)(y_{i+1} - y_i), \tag{6}$$

with weights

$$b_j(\beta) = (j+1)^{\beta} - j^{\beta}. \tag{7}$$

3 SIR model

The SIR model gives for Kermarck and Mckendrick in 1927 (KERMACK; MCKENDRICK, 1927), it is introduced by three nonlinear differential equations, which connects the number of the Susceptible, Infected, and Removed individuals in disease spreading over time, denoted by the variables $S(t), I(t), R(t)$, respectively. The sum $N = S(t) + I(t) + R(t)$ is an important property, because the variable N describes the total number of the population, that is constant. The system of the equations is given for:

$$\begin{cases} \frac{dS}{dt} &= -\frac{\beta I(t)S(t)}{N} \\ \frac{dI}{dt} &= \frac{\beta I(t)S(t)}{N} - \gamma I(t) \\ \frac{dR}{dt} &= \gamma I(t) \end{cases} \tag{8}$$

where β is the infection rate, γ is the recovery rate. $\frac{dS}{dt}$ is the rate of change in the number of susceptible over time, $\frac{dI}{dt}$ is the rate of variation of the infected over the time and $\frac{dR}{dt}$ is the rate of variation in the number of recovered over time.

The discrete form to system of equation Eq.8 is given by (KEELING; ROHANI, 2011):

$$\begin{cases} S_{n+1} &= S_n - \frac{\beta I_n S_n}{N}\Delta t \\ I_{n+1} &= I_n + \frac{\beta I_n S_n}{N}\Delta t - \gamma I_n \Delta t \\ R_{n+1} &= R_n + \gamma I_n \Delta t \end{cases} \tag{9}$$

The use of the SIR model to study the dynamics of COVID-19 is justified by its high performance in epidemiological problems, despite being a simple model, its results are very close to real problems. The unit of time used to study the evolution of the disease is given in days.

4 Fractional SIR model

In this section applies the Caputo fractional derivative to generalize the SIR model. This fractional model considers the memory effects in the real problem. The equation is given by:

$$\begin{cases} {}^{C}\mathcal{D}_t^{\alpha} S(t) &= -\frac{\beta I(t)S(t)}{N} \\ {}^{C}\mathcal{D}_t^{\alpha} I(t) &= \frac{\beta I(t)S(t)}{N} - \gamma I(t) \\ {}^{C}\mathcal{D}_t^{\alpha} R(t) &= \gamma I(t) \end{cases} \tag{10}$$

Based on (ALMEIDA, 2020; MAZORCHE; MONTEIRO, 2021; BARROS et al., 2021), we present the discrete fractional SIR model:

$$\begin{cases} {}^{C}\widehat{\mathcal{D}}^{\alpha} S_n &= -\frac{\beta I_n S_n}{N} \\ {}^{C}\widehat{\mathcal{D}}^{\alpha} I_n &= \frac{\beta I_n S_n}{N} - \gamma I_n \\ {}^{C}\widehat{\mathcal{D}}^{\alpha} R_n &= \gamma I_n \end{cases} \tag{11}$$

Discretizing the derivative of Caputo via the rectangle method, the variations are preserved and replicated in the following steps, thus respecting the so-called memory effect. Different from the discretization of the integer-order derivative where each variation is associated as one single step.

5 Epidemiologic Dynamics of COVID-19 in Maranhão, Brazil

COVID-19, caused by the SARS-COV-2 virus, has an extensive clinical spectrum ranging from asymptomatic infections to very severe conditions that can lead to death in a few weeks or days. According to the Brazilian Ministry of Health, the transmission of SARS-Cov-2 is given through one sick person to another or by close contact through sneezing, coughing, saliva droplets, and contaminated objects or surfaces. For the past two years, the whole world has had a relentless struggle against this disease. It causes several damages, mainly in the economic and health systems of all countries, without exception. Crowded hospitals and high unemployment prove how devastating the virus is.

The first official record of the disease was in Hubei Province, near Wuhan, China in December 2019. In Brazil, the first case was detected on February 26, 2020, and in the state of Maranhão on March 16, 2020. In order to learn about the dynamics of COVID-19, a survey of the number of infected cases in the State of Maranhão was made, based on data collected on the website of the State Secretariat of Maranhão, during 2020. With the available information, the application of these data in the SIR model was performed, considering the fractional and entire orders.

6 Numerical implementations

As stated earlier, this work aims at a better understanding of the spread of COVID-19. This is done by studying the dynamics of the number of secondary cases produced by an infected individual, that is, to determine the basic reproductive number R_0. This value is estimated by applying curve fitting between the SIR model (fractional and integer) and the the real data of infected by COVID-19 in Maranhão. As can be seen in figures 1 and 2. The values found for R_0 are shown in Tables 1-3 allowing the following considerations:

- The values to R_0 via fractional SIR model can be considered more realistic than classic integer SIR model. This is justified by the fact that the fractional model generates better approximations in relation to the real data.

- Due to the classical model having exponential behavior, there is a greater restriction for R_0 values than for the fractional case. Just look at the value of R_0 for the ten first days, because in the case of whole order the value greater than 6 to R_0, it is already impossible to approximations.

- The data of the accumulated infected is better to work with than the data of the daily infected. Because, the dispersion that appears in daily cases makes the analysis more complex.

- The Python (SAHA, 2015) programming languages is used to build the graphics and application in the curve fitting.

- The tables show a decrease in Ro values, which can be explained by the implementation of restrictive measures by the state government. Social distancing and the use of masks are examples of these restrictions.

Table 1: Values of R_0 obtained via fractional SIR model with $\alpha = 0.9$

Accumulated infected						
Period (days)	0 to 10	12 to 42	42 to 80	80 to 250	Remaining breaks	Average
R_0	9.66	4.9	2.62	1.04	1.22	3,888
α	0.9	0.9	0.9	0.9	0.9	
Daily Infected						
Period (days)	0 to 20	22 to 52	62 to 80	80 to 250	Remaining breaks	Average
R_0	9.66	5.46	2.59	1.04	1.22	3,994
α	0.9	0.9	0.9	0.9	0.9	

Table 2: Values of R_0 obtained via fractional SIR model with $\alpha = 0.95$

Accumulated infected						
Period (days)	0 to 20	22 to 45	60 to 80	80 to 250	Remaining breaks	Average
R_0	9.66	5.46	2.62	1.02	1.22	3,996
α	0.95	0.95	0.95	0.95	0.95	
Daily Infected						
Period (days)	0 to 8	12 to 42	42 to 80	80 to 250	Remaining breaks	Average
Ro	9.66	4.9	2.62	0.83	1.22	3.846
alpha	0.95	0.95	0.95	0.95	0.95	

Table 3: Values of R_0 obtained via classical SIR model with $\alpha = 1$

Accumulated infected						
Period (days)	0 to 15	15 to 30	30 to 63	105 to 150	Remaining breaks	Average
R_0	5.46	3.92	1.86	1.11	1.04	2.678
α	1	1	1	1	1	
Daily Infected						
Period (days)	0 to 18	30 to 50	53 to 70	85 to 250	Remaining breaks	Average
R_0	4.2	2.52	1.75	0.91	1.22	2.12
α	1	1	1	1	1	

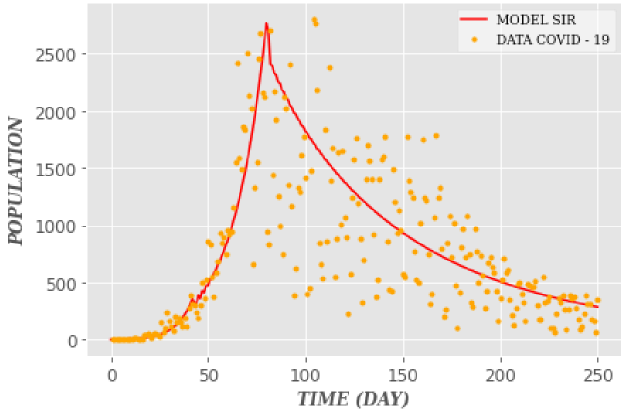

Figure 1: Daily cases and $\alpha = 0.9$

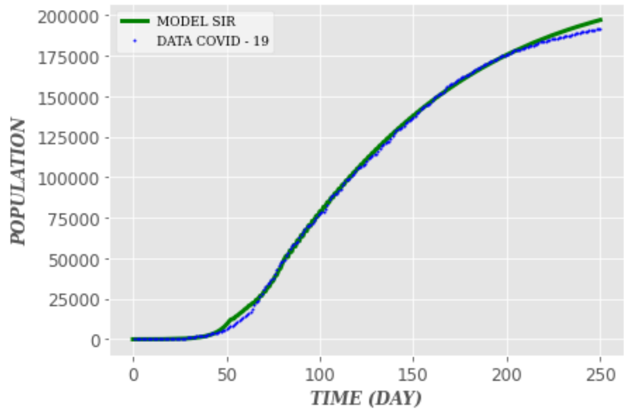

Figure 2: Cumulative cases and $\alpha = 0.9$

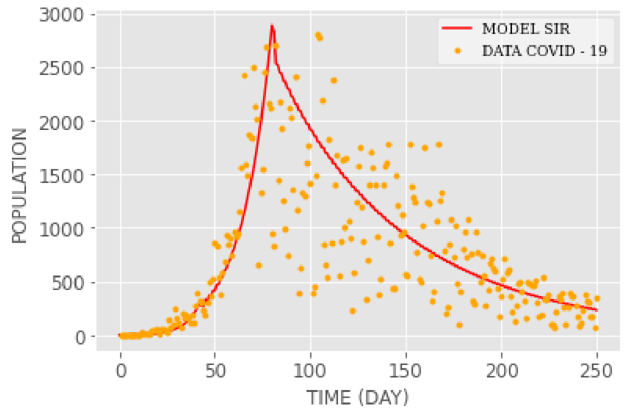

Figure 3: Daily cases and $\alpha = 0.95$

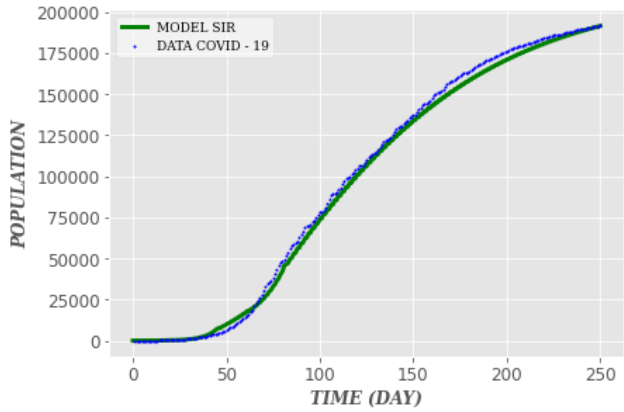

Figure 4: Cumulative cases and $\alpha = 0.95$

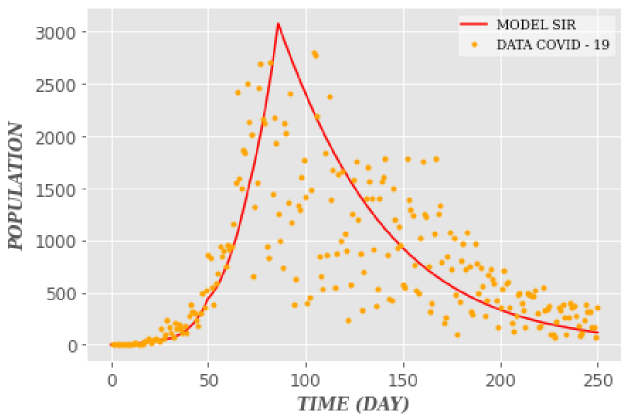

Figure 5: Daily cases and $\alpha = 1$

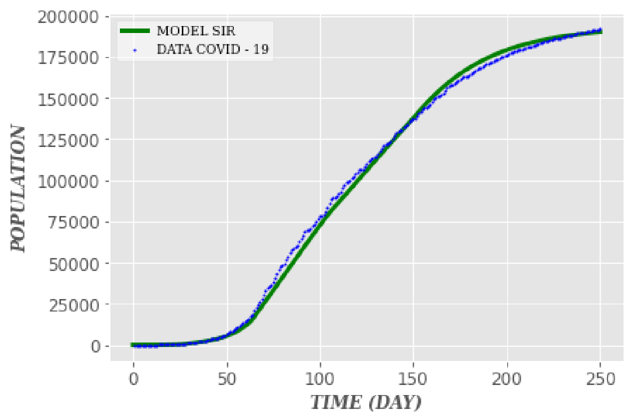

Figure 6: Cumulative cases and $\alpha = 1$

7 Concluding and remarking

The fractional model shows to be a good alternative as mathematical modeling for the study of epidemiological problems. Based on curve adjustments, this has proven to be more effective than the entire order model. The values of R_0 were closer to reality for the behavior of COVID-19, considering the non-integer order of α as can be observed in the tables and graphs. The values of R_0 are essential, as they allow a better understanding of the dynamics of the disease, allowing more efficient approaches against the pandemic. One can also highlight the level of complexity that appears when analyzing the number of daily infected, unlike the analysis for the number of accumulated infected. This difference is due to the dispersion of data for daily cases, implying the difficulty of curve adjustment. Finally, the results obtained reinforce how restrictive measures have a direct effect on the number of infected people.

References

ALMEIDA, R. Dynamical analysis of a fractional sir model with treatment and quarantine. **Chaotic Modeling and Simulation**, Athens, v. 2, p. 115–124, 2020.

BARROS, L. C. et al. The memory effect on fractional calculus: an application in the spread of covid-19. **Computational and Applied Mathematics**, Rio de Janeiro, v. 40, n. 3, p. 1–21, 2021.

COSTA, F. et al. Modelo sir aplicado na dinâmica da covid-19 no estado do maranhão, Brasil. **Revista de Matemática**, v. 1, n. 1, p. 18–34, 2021.

KEELING, M. J.; ROHANI, P. **Modeling Infectious Diseases in Humans and Animals**. Princeton: Princeton University Press, 2011.

KERMACK, W. O.; MCKENDRICK, A. G. A contribution to the mathematical theory of epidemics. **Proceedings of the royal society of london. Series A, Containing papers of a mathematical and physical character**, The Royal Society London, v. 115, n. 772, p. 700–721, 1927.

LI, C.; ZENG, F. **Numerical methods for fractional calculus**. Boca Raton: CRC Press, 2015. v. 24.

MAZORCHE, S. R.; MONTEIRO, N. Z. Fractional epidemiological models: what is lost, what is gained, what is transformed? **Proceeding Series of the Brazilian Society of Computational and Applied Mathematics**, v. 2, n. 1, 2021.

OLIVEIRA, E. C. **Solved exercises in fractional calculus**. Berlin: Springer International Publishing, 2019.

PŁOCINICZAK, Ł. On a discrete composition of the fractional integral and caputo derivative. **Communications in Nonlinear Science and Numerical Simulation**, Amsterdam, v. 108, p. 106234, 2022.

REZENDE, J. M. Epidemia, endemia, pandemia, epidemiologia. **Revista de Patologia Tropical/Journal of Tropical Pathology**, v. 27, n. 1, 1998.

SAHA, A. **Doing Math with Python**: use programming to explore algebra, statistics, calculus, and more! San Francisco: No Starch Press, 2015.