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Fractional calculus applied to evaluate stress concentration and shear effects in simply supported beams

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Abstract

Simply supported prismatic beams submitted to uniformly distributed static load are modeled in the ANSYS environment under linear elasticity limits. However, the beam deflection profile may behave in disagreement with the predictions from either Euler-Bernoulli (EB) or even Timoshenko-Ehrenfest (TE) linear theories, depending on the load magnitude and geometrical properties of the cross section. Careful examination on the ANSYS stress output data reveals, for those cases, local stress concentrations on the support contacts areas due to the intensive force reactions, which end up affecting the resulting transversal deflection along the entire beam, although the linear regime is ensured for all finite elements. Another way to verify these effects is to employ fractional calculus modeling on the EB or TE equations so that to confront the general fractional solutions with the ANSYS deflection outputs, which leads to a fractional order model whenever the stress concentration is present. The deviation of the fractional order with respect to the integer order of reference allows to measure the degree of relevance of the supporting reactions on the structure behavior. Both EB and TE theories lead to similar results, whose only difference relies on the shear effects already predicted analytically in the TE modeling, which may also be estimated on its magnitude by comparison using fractional calculus. Several case studies have been conducted and led to the development of a new estimate for structural analysis.

Keywords: fractional calculus, structure, Euler-Bernoulli, Timoshenko, Timoshenko-Ehrenfest, ANSYS

1 Introduction

Most of the widely used Finite Element Method (FEM) commercial software can represent the behavior of many physical systems with high accuracy (ANSYS), which is directly favored, for instance, with simplicity on the system geometry or boundary conditions, for example. This accuracy may allow the use of their results as if they were acquired from very accurate experiments, depending on the aforementioned simplicity. A simple system may have its simulated numerical behavior so close to reality that it could be used to explore or even discover physical phenomena implicitly participant in the software output. To explore such numerical results in this way, the detection of nonlinearities is useful, as the nonlinear behavior is a common feature in nature (ALVES FILHO, 2018). Fractional calculus is an approach for this. Taking ANSYS as a renowned (and therefore reliable) FEM software for the simulation of structures and taking simply supported prismatic beams as the needed simple systems, the prerequisites for employing fractional calculus are fulfilled.

Two beam theories such as Euler-Bernoulli (EB) and Timoshenko-Ehrenfest (TE) model simple beams under linear elastic constraints by means of integer order linear differential equations. The first one accomplishes small deflections of long beams with cross sections orthogonal to the neutral plane, while the second one dismisses the requirement of orthogonality, and therefore accomplishes the shear effects proportionally present with the beam thickness in the load direction. Hence TE modeling is more accurate than EB's.

Solving fractional models for both beam theories and taking the respective ANSYS outputs as the feeding information, the integer order differential equations end up transformed in fractional order ones. Depending on the deviation of the fractional order with respect to the integer order of reference, a quantitative measure of nonlinearity is established, whose causes become focus of investigation. Simply supported prismatic beams submitted to uniformly distributed static load are modeled in the ANSYS environment under linear elasticity limits. Dimensions and load are adjusted so that the beam deflection profile behaves in disagreement with the predictions from either EB or even TE linear theories. After careful examination on the ANSYS stress outputs for those cases, local stress concentrations are revealed on the support contacts areas due to the intensive force reactions, which end up affecting the resulting transversal deflection along the entire beam, although the linear regime is ensured for all finite elements.

Both EB and TE theories lead to similar results, whose only difference relies on the shear effects already predicted analytically in the TE modeling, which may also be estimated on its magnitude by comparison using fractional calculus. In the case of slender beams, fractional EB and fractional TE converge, as the remaining cause for nonlinearity is the stress concentration alone. Both thick and slender beams have been initially explored under EB and TE contexts in the works of (LUSTOSA; BANNWART; OLIVEIRA, 2021); however, this study did not consider the stress concentration effects, which has become our main focus the present investigation. Several case studies are conducted in this work towards the development of a new estimate for structural analysis.

2 Analytical development

The structure model chosen for the development of this study is a simply supported beam with rectangular cross section consisting of a linear elastic material subjected to an uniformly distributed loading $\vec{q}$ throughout its span. The bending moment, represented by M, is aligned with the principal moment of inertia in the horizontal direction. V represents the shear force.

According to (HIBBELER, 2010), the deflection in elastic regime is given by the equation

$$EI\frac{d^4v(x)}{dx^4} = -q(x), \tag{1}$$

where E is the Young's modulus and I is the rectangular moment of inertia of the cross section.

The model presented in Equation 1 is known as the Euler-Bernoulli model, since the hypotheses used to obtain it are equivalent to the EB theory hypotheses for small beam deflections. These hypotheses state that the initially flat cross sections remain flat after deformation. Furthermore, we have that the cross sections initially perpendicular to the barycentric axis, before bening, remain perpendicular after the deflection.

Solving Equation 1 with the appropriate conditions for the proposed problem, as (LUSTOSA; BANNWART; OLIVEIRA, 2021), we get

$$v(x) = \frac{q(x)}{EI}\left(\frac{-x^4}{24} + \frac{Lx^3}{12} - \frac{L^3x}{24}\right), \tag{2}$$

where L is the length of the beam.

2.1 Fractional Euler-Bernoulli solution

In recent times, the fractional calculus methodology has gained importance and greater visibility, being useful, for example, to explain the complex behavior of many phenomena, among others, we mention those resulting from viscoelasticity, (MAINARDI, 2010).

It is important to highlight that there are several types of approaches, in particular the type of formulation of non-integer order derivative, coined with the name of fractional derivative, (TEODORO; MACHADO; OLIVEIRA, 2019).

In this work, we use the formulation as proposed by Caputo that considers the fractional derivative as a fractional integral of an integer-order derivative (CAMARGO; OLIVEIRA, 2015) and (HERRMANN, 2011), for example. In what follows we focus on solving a non-integer order differential equation. The fractional model proposed for such a solution is the fractional ordinary differential equation of order α, similar to the Equation (1), given by

$$\frac{d^\alpha v(x)}{dx^\alpha} = -\frac{q(x)}{EI}, \tag{3}$$

with $4 \leq \alpha < 5$.

For $m \leq \alpha < m+1$ with α non-integer and m integer, the Laplace transform for derivatives of order α is given by (TEODORO, 2014)

$$L[f^{(\alpha)}(x)] = s^\alpha F(s) - \sum_{k=1}^{m}[s^{\alpha-k}f^{(k-1)}(0)], \tag{4}$$

where $F(s)$ is the Laplace transform of $f(x)$.

It is important to highlight that we can take the variation of α in the interval $m - 1 < \alpha \leq m$. Hence, the transform Equation (4) can be written in the form

$$L[f^{(\alpha)}(x)] = s^{\alpha}F(s) - \sum_{k=0}^{m-1}[s^{\alpha-1-k}f^{(k)}(0)]. \tag{5}$$

Applying Equation (4) in Equation (3), and using the boundary conditions as proposed in (LUSTOSA; BANNWART; OLIVEIRA, 2021), we obtain

$$v(x) = \frac{q(x)}{EI}\left[\frac{-x^{\alpha}}{\Gamma(\alpha+1)} + \frac{4L^{\alpha-3}}{2^{\alpha-1}\Gamma(\alpha+1)}(2^{\alpha-1}-\alpha)x^3 + \frac{L^{\alpha-1}}{2^{\alpha-1}\Gamma(\alpha+1)}(4\alpha - 3.2^{\alpha-1})x\right]. \tag{6}$$

Employing $\alpha = 4$ into Equation (6) and taking into account that for $n \in \mathbb{N}$ we have $\Gamma(n+1) = n!$, the integer solution is recovered in Equation (2).

Also according to (LUSTOSA; BANNWART; OLIVEIRA, 2021), we can deduce the fractional solutions for the bending moment M and the shear force V. Such solutions are given by.

$$M(x) = \frac{q(x)(\alpha-1)}{\Gamma(\alpha)}\left(-x^{\alpha-2} + L^{\alpha-3}x\right) \tag{7}$$

and

$$V(x) = \frac{q(x)(\alpha-2)(\alpha-1)}{\Gamma(\alpha)}\left(-x^{\alpha-3} + \frac{L^{\alpha-3}}{2^{\alpha-3}}\right), \tag{8}$$

respectively.

See that entire solutions are retrieved for $\alpha = 4$.

2.2 The shear effects of the Timoshenko-Ehrenfest beam theory

From the TE beam theory (ASSAN, 2010), for cross sections suffering small distortions, without losing linearity, the shear deformation is given by the first order ordinary differential equation

$$\frac{dv}{dx} = -c\frac{V}{GA}, \tag{9}$$

where $v = v(x)$, $V = V(x)$, G is the cross elasticity modulus, and c is the shear coefficient related to the bending deformation at the height of the section, which depends on its geometric dimensions and shape, (LUSTOSA; BANNWART; OLIVEIRA, 2021).

In Equation (9), A is the cross-sectional area, $G = \dfrac{E}{2(1+\nu)}$ for rectangular sections (our case study), and $c = \dfrac{12+11\nu}{10(1+\nu)}$, where ν is a Poisson module.

Solving this equation and using the superposition principle, according to (LUSTOSA; BANNWART; OLIVEIRA, 2021), the equation for the transversal deflection, including bending and shear, is given as follows:

$$v(x) = \frac{q(x)}{EI}\left(\frac{-x^4}{24} + \frac{Lx^3}{12} - \frac{L^3x}{24}\right) + \frac{q(x)c}{GA}\left(\frac{x^2}{2} - \frac{Lx}{2}\right). \tag{10}$$

The TE beam theory contemplates the effects dealt with in this section, which is discussed below.

In the development of the TE theory, the first hypothesis of the Euler-Bernoulli theory is preserved, that is, that the cross sections remain flat after the deformation; however, the second one is not considered, that is, the barycentric axis does not remain perpendicular to the cross section after deformation. Therefore, in TE theory, the cross section flexes and rotates, (FLEISCHFRESSER, 2012). The model of that theory is given by the following differential equation system:

$$\begin{cases} \dfrac{d^3\theta(x)}{dx^3} = \dfrac{q(x)}{EI}, \\ \dfrac{dv(x)}{dx} = \theta(x) - \dfrac{EIc}{GA}\dfrac{d^2\theta(x)}{dx^2}, \end{cases} \tag{11}$$

where the dependent variables of the $\theta = \theta(x)$ and $v = v(x)$ system represent, respectively, the cross section rotation and the deflection of the barycentric axis, therefore including the shear effects, (LUSTOSA; BANNWART; OLIVEIRA, 2021).

Solving the system for our case study considering the appropriate boundary conditions, as given in (LUSTOSA; BANNWART; OLIVEIRA, 2021), we obtain

$$v(x) = \frac{q(x)}{EI}\left(-\frac{x^4}{24} + \frac{Lx^3}{12} - \frac{L^3x}{24}\right) + \frac{cq(x)}{GA}\left(\frac{x^2}{2} - \frac{Lx}{2}\right) \tag{12}$$

Since Equation (12) matches Equation (10), we propose in this work a fractional solution for the TE model.

2.3 Fractional Timoshenko-Ehrenfest solution

Analyzing the development and results obtained in Subsection 2.2, the fractional TE model for our studied problem can be given by the superposition principle, that is,

$$v(x) = v_{FEB}(x) + v_{FC}(x), \tag{13}$$

where $v_{FEB}(x)$ and $v_{FC}(x)$ are the fractional solutions for deflection (effects of bending moments) and shear, respectively, (LUSTOSA; BANNWART; OLIVEIRA, 2021).

Solving such a model, we obtain

$$v(x) = A\left[-x^\alpha + \frac{4L^{\alpha-3}}{2^{\alpha-1}}(2^{\alpha-1} - \alpha)x^3 + \frac{L^{\alpha-1}}{2^{\alpha-1}}(4\alpha - 3.2^{\alpha-1})x\right] + B\left(x^{\alpha-2} - L^{\alpha-3}x\right), \tag{14}$$

where $A = \dfrac{q(x)}{EI\Gamma(\alpha+1)}$ and $B = \dfrac{cq(x)(\alpha-1)}{GA\Gamma(\alpha)}$.

The equation (14) is the fractional TE solution that provides the deflection suffered by a simply supported beam under an uniformly distributed load, (LUSTOSA; BANNWART; OLIVEIRA, 2021). For $\alpha = 4$, Equation (14) matches Equation (12) and the solution is verified.

3 Interpretation of physical effects using FEB and FTE fractional solutions

After deducting the fractional solutions Euler-Bernoulli (FEB) and Timoshenko-Ehrenfest (FTE), we now aim to use them to interpret physical effects that influence the deflection of a simply supported beam with distributed load, which are not contemplated by the classical integer solutions EB and TE. In this article, we focus on the influence of the stress variation caused by the supports at the beam ends and their possible effects on the deflection, also highlighting the first relevant fact observed, which is the contemplation of the shear effects by FEB. Later, we intend to investigate other physical effects, such as buckling, plastic deformations and other new ones that may arise during the investigations.

3.1 Interpreting the shear effects

Analyzing the obtained fractional solutions, the first noticeable relevant fact is that the FEB solution contemplates the shear effects previously contemplated only by the TE equation. To characterize this fact, we consider a thick beam made of AISI 1020 steel with dimensions $(b, h, L) = (0.3m; 1m; 5m)$ submitted to a distributed load with constant modulus given by $q = 5 \times 10^4 N/m$. Such a beam may be considered thick, as its aspect ratio is $\left(\frac{L}{h} = 5\right)$, and therefore the shear effects in the beam extremities are important. According to (SILVA, 2019), the shear effects become relevant for a aspect ratios lower than 10.

Confronting the maximum solutions of the FEB and TE equations, we have that the maximum solution FEB is taken in the maximum solution TE to $\alpha = 4.0964$. Applying this α into the generalized FEB equation, the particular FEB solution is obtained, which includes the shear effects.

The graph in Figure 1 shows the FEB solution for $\alpha = 4.0964$ confronted against the TE's. It can be noticed that the FEB solution contemplates the TE's throughout the range in which the beam is defined.

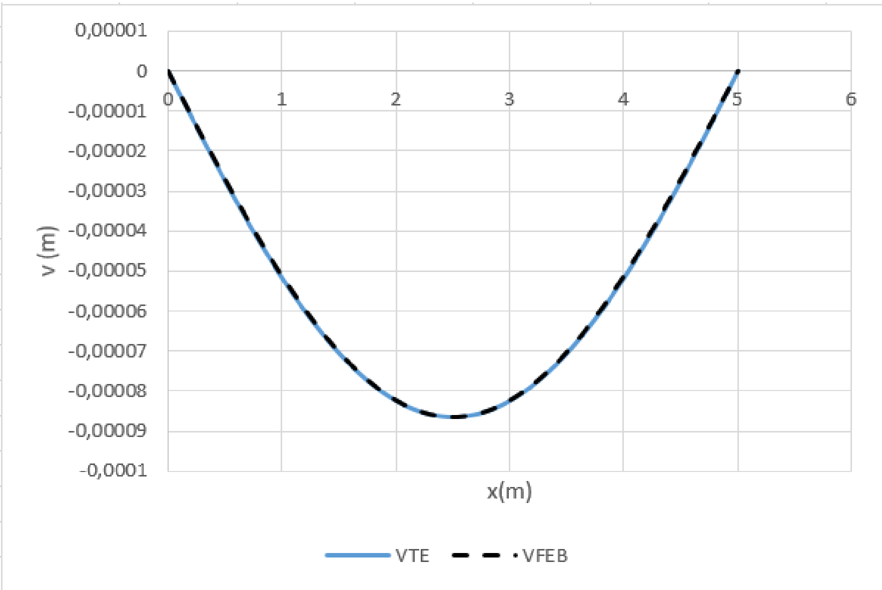

Figure 1: Confrontation between FEB and TE solutions.

To better evidence this fact, we determined the relative error E_r between the solutions FEB with $\alpha = 4.0964$ and TE's by defining E_r as being

$$E_r = \frac{|v_{TE} - v_{FEB}|}{|v_{TE}|}, \tag{15}$$

where v_{TE} is the deflection corresponding to the solution TE and v_{FEB} is the deflection FEB for $\alpha = 4.0964$. Finally, we calculate the average error E_m, defined as the arithmetic average of the elements of E_r, obtaining $E_m = 0.7\%$. To evaluate the significance of such an error, the actual context of use of the structure shall be pondered. Either way, this value shows that the fractional calculus approximation does not exactly reproduce the solution of reference; but it may be used, generally speaking, as a tool to improve simpler analytical models. It must be said we have verified that such a difference has not come from error associated with the numerical rounding made by the computational software during the execution operations.

3.2 Interpreting support effects on a thick beam

In simply supported beams, the stresses present in the supports are much more intense than the stresses produced by bending or shear moments,(TIMOSHENKO; JAMES, 1983). These stress concentrations can cause bending variation, even when the beam is subjected to low stresses, (LI; LEE, 2015). We aim then to investigate these facts and interpret them, using the generalized fractional solutions FEB and TFE.

To exemplify such facts, we take a thick beam made of AISI 1020 steel with dimensions $(b, h, L) = (0.3m; 1m; 5m)$, subjected to a specific load, whose respective stresses are lower than linear limits, which characterize the small deflection regime for the material used. In this case, we analyze the deflection profile in the linear and nonlinear simulations in the ANSYS software. For $q = 3 \times 10^4 N/m$, the maximum yield stress for such a load is $\sigma = 9.6247 \times 10^6 Pa$, while the maximum yield stress for AISI 1020 steel varies between $\sigma = 2.5 \times 10^8 Pa$ and $\sigma = 4.7 \times 10^8 Pa$, (LEITE et al., 2017) and (TIMOSHENKO; JAMES, 1983).

We performed linear and nonlinear simulations for such a load calculating the average error E_m between the deflection obtained, analogously to what was done in Section 3.1.

From this we get the corresponding average error $E_m = 0.0053\%$. Hence, we conclude that the deflections in both analyzes present considerable approximation throughout the interval in which the beam is defined.

The approximation of deflections implies approximation of stresses in both analyses. This fact is proven when we calculate the average error between the stresses obtained in the respective simulations, as we obtain $E_m = 0.0027\%$.

The approximation of stresses implies equality in the support reactions, a fact proven in Table 1, where we extract the support reactions from both simulations.

Table 1: Supports reactions.

	linear simulation	nonlinear simulation
fixed support	$7.5000 \times 10^4 N$	$7.5000 \times 10^4 N$
mobile support	$7.5000 \times 10^4 N$	$7.5000 \times 10^4 N$

The most important fact that we must highlight here is that the maximum concentration of stresses is on the supports. Figure 2 obtained from the ANSYS simulation bring this fact out.

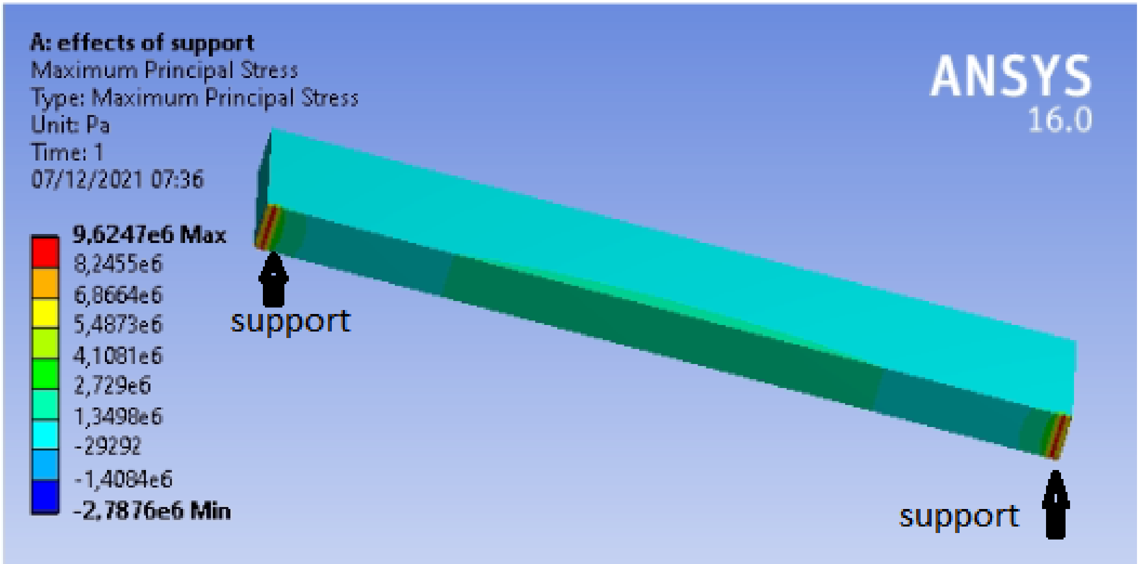

Figure 2: Representations of the stresses in the beam.

Looking at Figure 2, we can see that the maximum stress concentration is located on the support regions. From the stress legend, the highest concentrations are shown on the ares highlighted in red color, which is exactly where the supports react.

The next step is to confront the generalized fractional deflections FEB and FTE with the deflection obtained from the linear ANSYS simulation. Performing this procedure, we notice that the deflection for linear ANSYS solution is larger than the deflection for TE solution. This suggests that the linear ANSYS solution contemplates, in addition to shear, another physical effect. We deduce that this effect is caused by the supports because, as shown in Figure 2, the highest concentration of stresses present in the beam are precisely at these points.

By fitting the curves, using as reference the maximum solution provided by ANSYS, the FEB and FTE solutions are taken at this maximum to $\alpha = 4.3115$ and $\alpha = 4.1811$, respectively. Hence, taking the ratios $\frac{v_{EB}}{v_{AN}}$, $\frac{v_{TE}}{v_{AN}}$, $\frac{v_{FEB}}{v_{AN}}$ and $\frac{v_{FTE}}{v_{AN}}$,we get the graph in Figure 3 as follows.

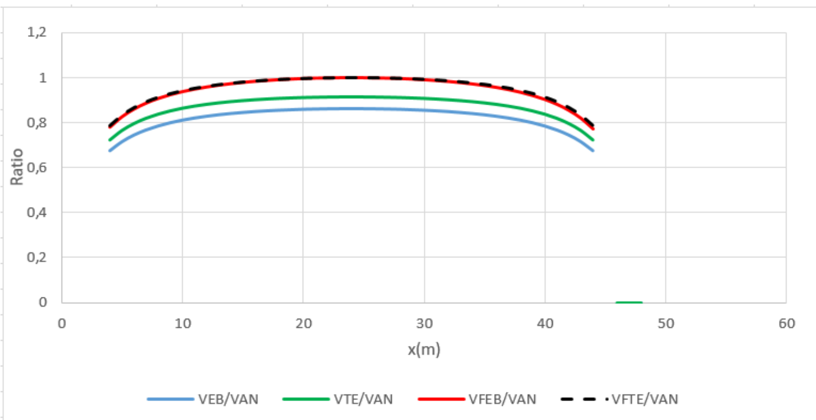

Figure 3: Ratio between EB, TE, FEB and FTE fractional solutions in relation to ANSYS solution.

Analyzing the graph in Figure 3, we see that the FEB and FTE analytical solutions include the ANSYS solution in most of the range in which the deflection is defined, with the FTE solution presenting itself slightly better. Thus, we have the effects of the supports on the deflection, being considered by these solutions with relevant proximity to the real situation, which we are representing by the ANSYS numerical solution. As observed in the graph in Figure 3, this fact does not occur for the analytical solutions EB and TE.

The great relevance to be highlighted in this case is that the respective fractional solutions are able to detect the effects of the supports on the deflection and, in addition, they contemplate with considerable proximity to the real solution. It is worth noting that such effects are detected by the solutions mentioned with the order α, different from the entire order, and therefore nonlinear, as it is their way of accommodating the referred effects. By the ANSYS software, these effects were detected in the linear simulation, due to the entire characterization of the problem being developed in linear regime.

4 Slender beam analysis

In the slender case, we analyze the cases of small as well as large deflections. The justification for analyzing the case of large deflections is to try to investigate whether when the interference of the supports has little influence on the deflection, the action of some other physical phenomenon may arise. If so, we interpret using the FEB and FTE fractional models.

4.1 Effects of supports on a slender beam subjected to small deflections

To analyze the support effects in the case of a slender beam subjected to low stresses (small deflections), we take the beam with dimensions $(b, h, L) = (0.3m; 0.1m; 5m)$. In this type of beam, the shear effects are disregarded, as they have a high aspect ratio. Due to this fact, we have that the EB and TE analytical solutions coincide. Making the linear and nonlinear simulations with load $q = 3 \times 10^2 N/m$, whose yield stress is $\sigma = 1.8754 \times 10^6 Pa$ and therefore within the linear limits, we noticed that the deflections coincide as in the previous analysis, because calculating the average error between the deflection of the respective simulations, we obtain $E_m = 0.18\%$. An interesting fact in this case is that the ANSYS solution coincides with the EB and TE solutions. Table 2 shows the respective average errors between the deflections of the solutions taken two by two.

Table 2: Average error between solutions.

compared deflections	average error E_m
EB-TE	0.1000%
EB-AN	0.0540%
TE-AN	0.0510%

From this, we conclude that there is no presence of nonlinear physical effects in these simulations, as the numerical solution extracted from ANSYS is contemplated by the EB and TE solutions, which are actually a particular case of the FEB and FTE solutions for the entire order ($\alpha = 4$). Therefore, we can conclude that in this beam configuration, whose stresses were increased below the yield stress, there is no significant influence of the support reactions. But when we do not have a significant influence from the supports, it is noticed that the highest stress levels are located in the central region of the beam. Hence, we thought of analyzing the case where the stresses are increased in a neighborhood of the yield point. That's what we'll do in the next section.

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4.2 Support effects on a slender beam subjected to large deflections

Now, to analyze the effects on the slender beam for the case of large deflections, we take the load $q = 3\times10^5 N/m$. The first fact that we noticed is that the support reactions decrease in the nonlinear simulation, as shown in Table 3.

Table 3: Support reactions.

	linear simulation	nonlinear simulation
fixed support	$7.5000\times10^5 N$	$7.3245\times10^5 N$
mobile support	$7.5000\times10^5 N$	$7.3245\times10^5 N$

From Table 3, it is clear that the stresses in nonlinear simulation decrease compared to linear simulation. This in fact occurs because taking the maximum stresses for the linear and nonlinear simulations we get $\sigma = 1.8754\times10^9 Pa$ and $\sigma = 1.7730\times10^9 Pa$. Consequently, we expect the deflection in the nonlinear simulation to decrease. To verify this, we analyze the linear and nonlinear simulations for the respective load and take the average deflection for the respective cases, obtaining respectively $f_{ml} = 0.2957$ and $f_{mnl} = 0.2764$, where f_{ml} and f_{mnl} are the average deflections for the linear and nonlinear simulations, respectively. Hence, we found that the deflection of the nonlinear simulation actually decreased, so we have a physical effect not covered by the linear simulation.

Note that taking the average error between the deflections of the linear simulation and the TE analytical solution, we obtain $E_m = 0.06\%$, which guarantees that the linear simulation, in this case, does not include effects other than those caused for the moments, because EB=TE. Furthermore, the highest stress levels are located in the central region of the beam, as we had already suspected. Thus, taking these facts into account, we can conclude that the physical effect contemplated in the deflection of the nonlinear simulation does not have a significant portion related to the supports.

Researching about such physical effect, we found similar numerical simulations in (ALVES FILHO, 2018) and, according to the reference, this effect is a consequence of the axial forces that arise from the tendency to approach the ends of the beam, as it is being subjected to loads that cause large deflections. Due to the nonlinearity of the phenomenon, the axial forces that arise are not proportional to the displacements, so there is a variation in stiffness, in this case, the variation is in geometric stiffness, which is increasing, as these axial forces that arise, cause the stiffening of the central lower part of the beam, as explained by (SORIANO, 2009). Due to this, the variation in the deflection is identified.

Finally, we can again use the FEB and FTE fractional analytical solutions to interpret this physical effect, as we have already highlighted before, it is the main objective of the work.

Confronting the respective fractional solutions with the ANSYS solution, we see that the maximum deflection of the fractional solutions are taken into the maximum deflection of the ANSYS solution when we take $\alpha = 3.8984$. Note that in this case, we have α less than the entire order, which is $\alpha = 4$. Therefore, we are considering the Laplace transform as defined in Equation 5.

Taking the value of α in the generalized solutions we find the particular solutions FEB and FTE for the deflection in the entire interval in which the beam is defined.

Once again, the ratios $\frac{v_{EB}}{v_{AN}}$, $\frac{v_{TE}}{v_{AN}}$, $\frac{v_{FEB}}{v_{AN}}$ and $\frac{v_{FTE}}{v_{AN}}$, we get the graph in Figure 4.

Analyzing the graph in Figure 4, we can see that the FEB and FTE fractional analytical solutions include the ANSYS solution in practically the entire interval in which the beam is defined. Therefore, we conclude that these solutions contemplate the physical effect detected by the nonlinear ANSYS

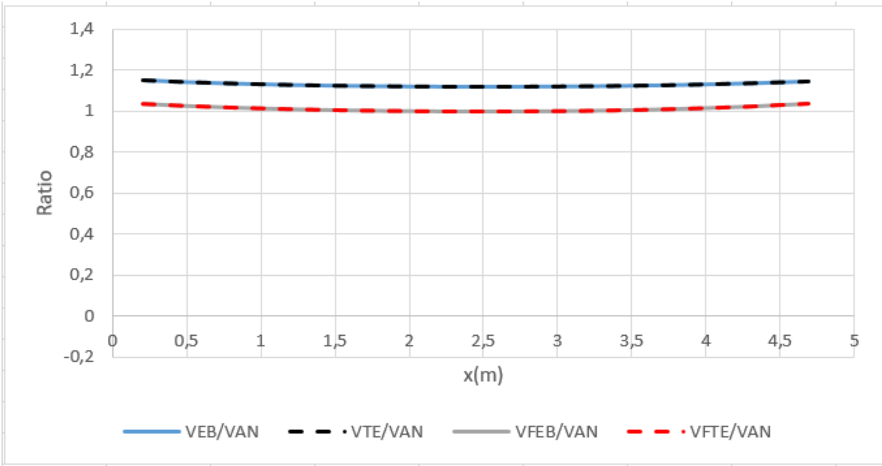

Figure 4: Ratio between fractional solutions EB, TE, FEB and FTE in relation to the ANSYS solution.

simulation, also characterizing its nonlinearity, since the α that fits the solution is different from the integer order.

We conclude the analysis by highlighting that the great relevance of the FEB and FTE fractional equations observed so far is to be able to describe the solution to the problem with much greater proximity to reality, a fact observed in the analyzed results, in addition to being highly efficient in detecting nonlinear phenomena. So, we can now say that we have analytical solutions that are able to contemplate possible effects of the supports caused on the deflection, due to the concentration of high stresses in these points, with a considerable approximation to reality and, furthermore, characterizing its nonlinearity.

Other physical effects were also considered, such as the one found in the case of the slender beam subjected to stresses close to the yield point, which is a consequence of the emergence of axial forces due to the tendency to approach the ends of the beam. These effects are not considered by the analytical solutions EB and TE, as the models from which such solutions are obtained considering the initial condition of the linearized structure.

The fact that the FEB and FTE models are able to include certain physical effects, as explained in this paper, is precisely because they have the memory effect loaded by the α factor, when it is different from the entire order, (BARROS et al., 2021) and (TEODORO; OLIVEIRA; OLIVEIRA, 2017). This feature, the integer solutions EB and TE do not have, as they are obtained from the generalized solutions FEB and FTE for $\alpha = 4$(integer order). In our study, the memory effect present in the FEB and FTE fractional solutions allows us to use them to interpret physical effects both in a linear and nonlinear condition.

5 Conclusions

In this work, we use the fractional solutions FEB and FTE to interpret physical effects that arise due to concentrated support reactions on simply supported beams subjected to a uniformly distributed loading, which are not covered by the integer solutions EB and TE. We begin by showing that the FEB fractional solution contemplates the shear effects previously only contemplated by the TE solution. Continuing the study, we used the numerical solution provided by the ANSYS software as reference and source of data to represent approximate real situation, as it, like other commercial software, is able to represent the behavior of many physical systems very accurately. Then, observing the stress variation due to the support reactions in the simulations carried out in ANSYS, we analyzed the influence of the effects of these reactions on the beam deflection and its respective characterization by the fractional solutions FEB and FTE. For the analysis of the respective physical effects, we employed examples of either thick or slender beams. For all these cases we have shown that the aforementioned solutions contemplate those effects with a relevant approximation to reality, presenting themselves much better than the classic EB and TE models. As the fractional solutions used are new, we see through this case study that they allow the development of new estimates about the various physical effects present in structural analysis of beams. In future works, we intend to use these fractional modelings as an estimate to study phenomena such as buckling, plastic deformations and others that may arise during the investigation.

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