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Numerical algorithms considering a dimensional correction parameter on the fractional order diffusion equation

Jhoab Pessoa de Negreiros
UERJ - Universidade do Estado do Rio de Janeiro
jhoab.negreiros@pos.ime.uerj.br

Carlos Antonio de Moura
Instituto de Matemática e Estatística
UERJ - Universidade do Estado do Rio de Janeiro
demoura@ime.uerj.br

Cristiane Oliveira de Faria
Instituto de Matemática e Estatística
UERJ - Universidade do Estado do Rio de Janeiro
cofaria@ime.uerj.br

Abstract

We deal with a generalization for the constant coefficient one-dimensional fractional diffusion equation by inserting the dimensional correction parameter τ in the model, as proposed in (GÓMEZ-AGUILAR et al., 2012). The fractional time variation is simulated by the Riemann-Liouville, Caputo-Fabrizio and Katugampola derivatives in order to compare each of these operators behavior. Numerical results drawn for the dimensionalized fractional diffusion equation are also compared with those obtained for the same equations when the dimension constraint was not taken into account.

Keywords: Fractional diffusion equation. Dimensional correction. Finite differences approximation. Riemman-Liouville, Caputo-Fabrizio and Katugampola fractional derivatives.

1 Introduction

In the 17th century, Leibniz and Newton independently developed the differential calculus considering integer order. These mathematical developments were inspired by phenomena as the movement of bodies and their variation. It reached sufficient rigor already in the 19th century. During this same period, the so-called non-integer or simply fractional-order calculus begins with the correspondence exchanged in 1695 between l'Hôpital and Leibniz for the case of the derivative of order 1/2. In general, fractional-order derivatives do not have a geometric interpretation with the simplicity as integer derivatives do. However, fractional derivatives allow the modeling of problems that involve concepts of non-locality and the memory effect and, consequently, in some cases it allows to describe systems with more elaborated aspects (CAMARGO; OLIVEIRA, 2015).

As the $19th$ century took off, many authors have also contributed to the fractional calculus research and emerging applications to problems in several areas of science so that its importance had steadily increased. Such development has led to an increasing number of formulations for the fractional derivative. In this way, the appropriate choice of the most convenient fractional derivated for each modeling requires careful analysis for the operators, so as to be able to deal with the most required aspects of the phenomena. It should be emphasized that such a task lacks to be restricted to the replacement of an integer order derivative by a decided fractional-order one. The modeling also requires taking care of the correct dimension equations (GÓMEZ-AGUILAR et al., 2012). In this way, a number of adjustments must be carried over.

This article bears the following organizational pattern. We first exhibit the definitions for the fractional derivatives known as Riemann-Liouville's, Caputo-Fabrizio's, and Katugampola's. Then we consider the linear diffusion equation with fractional derivative, being aware of its correct dimension. We close the present work by comparing numerical results drawn for the three different fractional operators chosen.

2 Fractional Operators

In 1974, in (ROSS, 1975) Ross proposed a five properties criterion which an operator should fulfill in order to be considered as a fractional derivative. Ortigueira and Machado, in 2015, rewrote this criterion (ORTIGUEIRA; MACHADO, 2015), adding the need for a fractional derivative to generalize Leibniz' rule. The five properties thereby proposed are:

1 The fractional derivative is a linear operator;

2 The order zero fractional derivative of a function is the function itself;

3 Whenever its order is a positive integer $n \in \mathbb{N}$, the fractional derivative must give rise to the same result as the n-th order common derivative, while for a negative integer $-n$, $n \in \mathbb{N}$, a similar generalization must be fulfilled;

4 The exponents law, $D^{\alpha} D^{\beta} f(x) = D^{\alpha+\beta}$, holds for $\alpha < 0$ e $\beta < 0$;

5 Leibniz' rule generalization also holds, namely,

$$D^{\alpha}\left(f(x)g(x)\right) = \sum_{k=0}^{\infty} \binom{\alpha}{k} D^{k} f(x) D^{\alpha-k} g(x),$$

where
$$\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}.$$

In her doctorate thesis, Teodoro (TEODORO, 2019) has classified the operators at the time well established in the literature in three different classes, namely, classical derivatives, local derivatives, and non-singular kernel derivatives. Afterward, it was verified whether each of these operators may be classified as a fractional derivative according to Ortigueira and Machado[1].

Our choice for the Riemann-Liouville, Caputo-Fabrizio and Katugampola fractional operators applied to the diffusion equation was taken in order to obtain results for one operator in each of the classes proposed by Teodoro.

2.1 Riemann-Liouville Fractional Derivative

Nikoly Sonin, in 1869, published the first work which has made use of the operator now mentioned as Riemann-Liouville derivative. This operator definition requires the concept of the fractional integral:

Definition 2.1 *For $\alpha > 0$ define the Riemann-Liouville fractional integrals on the left and right for a Riemann integrable function f on (a, b), respectively by*

$$_{RL}I^{\alpha}_{a,t}f(t) := \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s)ds\ , \tag{1}$$

$$_{RL}I^{\alpha}_{t,b}f(t) := \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1} f(s)ds\ , \tag{2}$$

where $\Gamma(\cdot)$ represents the gamma function.

The definition of the fractional derivative for an arbitrary order is based on the fraction integral definition, coupled to the fact that derivation and integration are inverse operators of each other (CAMARGO; OLIVEIRA, 2015).

Definition 2.2 *The Riemann-Liouville fractional derivatives of order $\alpha > 0$ on the left and right for a funtion $f(t)$, $t \in (a, b)$, are defined respectively as*

$$_{RL}D^{\alpha}_{a,t}f(t) := \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_a^t (t-s)^{m-\alpha-1} f(s)ds\ , \tag{3}$$

$$_{RL}D^{\alpha}_{t,b}f(t) := \frac{(-1)^m}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_t^b (s-t)^{m-\alpha-1} f(s)ds\ , \tag{4}$$

being m the positive integer for which $m-1 < \alpha \leq m$, called herein the integer roof for α.

[1](TEODORO, 2019) claims that this criterion choice is justified since it is a more restrictive choice than the one proposed by Ross.

2.2 Caputo-Fabrizio Fractional Derivative

In 2015 Michele Caputo and Mauro Fabrizio proposed a new definition for the fractional derivative with a non-singular kernel (CAPUTO; FABRIZIO, 2015). This operator, associated to a fractional derivative, is called the Caputo-Fabrizio derivative.

Definition 2.3 *For* $0 < \alpha < 1$*, the Caputo-Fabrizio derivative of order* α *is defined by*

$$ {}_{CF}D^{\alpha}_{a,t} f(t) := \frac{M(\alpha)}{1-\alpha} \int_a^t exp\left(-\alpha \frac{t-s}{1-\alpha}\right) f'(s) ds \tag{5}$$

where $t \geq 0$ *e* $M(\alpha)$ *is a normalization function for which* $M(0) = M(1) = 1$.

2.3 Katugampola Fractional Derivative

Before Caputo and Fabrizio, in 2014, Katugampola proposed a new formulation for the fractional derivative (KATUGAMPOLA, 2014). Katugampola derivative for arbitrary order differs from both Riemann-Liouville and Caputo-Fabrizio ones because it is a local operator defined in terms of a limit process.

Definition 2.4 *Let* $t \in \mathbb{R}$*,* $n \in \mathbb{N}$ *and* $f(t)$ *be a real function* n *times differentiable. Katugampola fractional derivative of order* α*, being* $n < \alpha \leq n+1$*, for the function* f *is given by*

$$ {}_{K}D^{\alpha} f(t) := \lim_{\epsilon \to 0} \frac{f^{(n)}(te^{\epsilon t^{n-\alpha}}) - f^{(n)}(t)}{\epsilon}, \tag{6}$$

for all $t > 0$.

For $\alpha \in (n, n+1]$, with $n \in \mathbb{N}$ and being f $(n+1)$ times differentiable in a point $t > 0$, then (6) may be written as

$$ {}_{K}D^{\alpha} f(t) = t^{1-\alpha+n} f^{(n+1)}(t). \tag{7}$$

3 Fractional Diffusion Equation

The classical problem related to the integer order diffusion equation consists in the search for a function $u(x,t)$, defined for $t \geq 0$ and $0 \leq x \leq L$, such that

$$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad 0 \leq x \leq L, \quad t \geq 0 \tag{8a}$$

$$u(x,0) = \phi(x), \quad 0 \leq x \leq L \tag{8b}$$

$$u(0,t) = l(t), \quad t \geq 0 \tag{8c}$$

$$u(L,t) = r(t), \quad t \geq 0\,, \tag{8d}$$

being known the functions $\phi(x)$, $l(t)$, $r(t)$ and $f(x,t)$. The parameter K denotes the considered body thermal diffusivity, and it depends on the thermal condutivity κ, on the density ρ and the material specific heat s, i.e., $K = \kappa/\rho s$ which has $[K] = x^2/t$ as its dimension, cf. (CANNON, 1984).

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Whenever the space derivative in this equation is replaced by a fractional derivative, say the Riemann-Liouville one, we get

$$\frac{\partial u}{\partial t} = \mathcal{K}_\alpha \;\; {}_{RL}D^{\alpha}_{0,t} \left(\frac{\partial^2 u}{\partial x^2} \right) + f(x,t), \tag{9}$$

for $0 < \alpha < 1$. This relation is called the fractional diffusion equation, being $\mathcal{K}_\alpha := \tau^\alpha K$ the fractional diffusion coefficient (LI; ZENG, 2015). Equation (9) coupled to the functions $\phi(x)$, $l(t)$ and $r(t)$ compose the initial value and boundary problem in the domain $D := \{(x,t)|\ 0 \le x \le L,\ t \ge 0\}$.

The change of the integer order model (8a) to any other of arbitrary order (9) causes a non-balance within the set of dimensions and unities associated to the original model. We thus, following (GÓMEZ-AGUILAR et al., 2012) and (GÓMEZ-AGUILAR; RAZO-HERNÁNDEZ; GRANADOS-LIEBERMAN, 2014), apply to (8a) the time fractional derivative coupled to a new parameter τ, whose unity is time, and then take the fractional derivative with the correction factor $\frac{1}{\tau^{-\alpha}} \frac{d^\alpha}{dt^\alpha}$, so that the fractional equation (9) now preserves dimension consistency.

4 Numerical Procedures

We have chosen the finite difference method to perform some numerical tests on (9), having applied a scheme based on backward Euler to deal with the three fractional operators. The computational mesh in the domain for (9) was defined by

$$x_i := i\Delta x, \quad i = 0,1,2,\ldots,M, \quad L = M\Delta x \ ,$$

$$t_n := n\Delta t, \quad n = 0,1,2,\ldots,N, \quad T = N\Delta t \ .$$

At the points (x_i, t_n) in the mesh we denote $u(x_i,t_n) \approx U_i^n$, having employed for the first time derivative the backward differences (10), while the space centered differences (11) have been taken for the second order space derivative in (9):

$$\partial_t^- U_i^n := \frac{u(x_i,t_n) - u(x_i, t_n - \Delta t)}{\Delta t} = u_t(x_i,t_n) - O\left(\frac{\Delta t}{2!}, u_{tt} \right), \tag{10}$$

$$\partial_x^2 U_i^n := \frac{u(x_i - \Delta x, t_n) - 2u(x_i,t_n) + u(x_i + \Delta x, t_n)}{h^2} = u_{xx}(x_i,t_n) + O\left(\frac{\Delta x^2}{12}, u_{xxxx} \right) \ . \tag{11}$$

Additional details can be found in (LEVEQUE, 2007).

4.1 Riemann-Liouville fractional diffusion equation

Grünwald-Letnikov fractional derivative for a sufficiently regular function $u(t)$ is equivalent to its fractional derivative in Riemann-Liouville sense (BALEANU et al., 2012). This fact makes it possible to get approximations for the Riemann-Liouville fractional derivative if we employ Grünwald-Letnikov's, by what we mean

$${}_{RL}D^{\alpha}_{0,t} u(t)\Big|_{t=t_n} \approx \frac{1}{\Delta t^\alpha} \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} u(t_{n-k}). \tag{12}$$

Expression (12) is a linear (order 1) approximation for any $\alpha > 0$. Nevertheless, for $1 < \alpha < 2$, it may give rise to unstable schemes (MEERSCHAERT; TADJERAN, 2004). By aplying the approximations (10) and (11) to (9), we get

$$\frac{U_i^n - U_i^{n-1}}{\Delta t} = \mathcal{K}_\alpha \, {}_{RL}D_{0,t}^{\alpha} \left(\frac{U_{i-1}^n - 2U_i^n + U_{i+1}^n}{\Delta x^2} \right) + f(x_i, t_n). \tag{13}$$

If we now apply the scheme (12) to the fractional derivative in (13), for $0 < \alpha < 1$, we reach

$$\frac{U_i^n - U_i^{n-1}}{\Delta t} = \frac{\mathcal{K}_\alpha}{\Delta t^\alpha} \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} \left(\frac{U_{i-1}^{n-k} - 2U_i^{n-k} + U_{i+1}^{n-k}}{\Delta x^2} \right) + f(x_i, t_n) \tag{14}$$

$$U_i^n = U_i^{n-1} + r \sum_{k=0}^{n} \omega(k) (U_{i-1}^{n-k} - 2U_i^{n-k} + U_{i+1}^{n-k}) + \Delta t f_i^n, \tag{15}$$

with $r := \frac{\mathcal{K}_\alpha \Delta t^{1-\alpha}}{\Delta x^2}$ e $\omega(k) := (-1)^k \binom{\alpha}{k}$. For $1 \leq i \leq M-1$, use of the matrix form to (15) gives

$$\boldsymbol{U^n} = \boldsymbol{U^{n-1}} + \sum_{k=0}^{n} \omega(k) \left(A\boldsymbol{U^{n-k}} + \boldsymbol{C^{n-k}} \right) + \Delta t \boldsymbol{f^n}, \tag{16}$$

or

$$(\boldsymbol{I} - A)\, \boldsymbol{U^n} = \boldsymbol{U^{n-1}} + \sum_{k=1}^{n} \omega(k) \left(A\boldsymbol{U^{n-k}} + \boldsymbol{C^{n-k}} \right) + \boldsymbol{C^n} + \Delta t \boldsymbol{f^n}, \tag{17}$$

where $\boldsymbol{f^n} = \left(f_1^n, f_2^n, \ldots, f_{N-1}^n \right)^T$, $\boldsymbol{U^n} = \left(U_1^n, U_2^n, \ldots, U_{N-1}^n \right)^T$, $\boldsymbol{C^n} = \left(rU_0^n, 0, \ldots, 0, rU_N^n \right)^T$ and

$$A := \begin{pmatrix} -2r & r & 0 & \ldots & 0 \\ r & -2r & r & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & r & -2r & r \\ 0 & \ldots & 0 & r & -2r \end{pmatrix}.$$

4.2 Caputo-Fabrizio fractional diffusion equation

We have chosen the approximation proposed by (QURESHI; RANGAIG; BALEANU, 2019) to deal with Caputo-Fabrizio fractional diffusion equation. The employed scheme relies on the use of the classical finite difference two-point formula for first order derivative,

$${}_{CF}D_{0,t}^{\alpha} u(t)\Big|_{t=t_n} = \frac{M(\alpha)}{\alpha \Delta t} \left(e^{\frac{\alpha}{1-\alpha}\Delta t} - 1 \right) \sum_{k=1}^{n} (u_{n-k+1} - u_{n-k})\, e^{-\frac{\alpha}{1-\alpha} k \Delta t}, \tag{18}$$

so as to reach a numerical method with first order local precision.

Through replacement of the Riemann-Liouville fractional derivative by Caputo-Fabrizio in (9) with posterior application of the approximations (10) and (11), we reach

$$\frac{U_i^n - U_i^{n-1}}{\Delta t} = \mathcal{K}_\alpha \, {}_{CF}D_{0,t}^{\alpha} \left(\frac{U_{i-1}^n - 2U_i^n + U_{i+1}^n}{\Delta x^2} \right) + f(x_i, t_n). \tag{19}$$

We then apply the numerical scheme (18) to (19), for $0 < \alpha < 1$, so as to get

$$\partial_t^- U_i^n = \frac{\mathcal{K}_\alpha M(\alpha)}{\alpha \Delta t} \left(e^{\frac{\alpha}{1-\alpha}\Delta t} - 1 \right) \sum_{k=1}^{n} \left(\partial_x^2 U^{n-k+1} - \partial_x^2 U^{n-k} \right) e^{-\frac{\alpha}{1-\alpha} k \Delta t} + f(x_i, t_n). \tag{20}$$

In the sequel we take the amortization function as $M(\alpha) = 1$. For $1 \leq i \leq M - 1$, the matrix form associated to (20) is

$$\boldsymbol{U^n} = \boldsymbol{U^{n-1}} + \sum_{k=1}^{n} \mathcal{E}(k) \left[A(\boldsymbol{U^{n-k+1}} - \boldsymbol{U^{n-k}}) + \boldsymbol{C^{n-k+1}} - \boldsymbol{C^{n-k}} \right] + \Delta t \boldsymbol{f^n} \tag{21}$$

or

$$\begin{aligned} (\boldsymbol{I} - \mathcal{E}(1)B)\boldsymbol{U^n} &= (\boldsymbol{I} - \mathcal{E}(1)B)\boldsymbol{U^{n-1}} + \sum_{k=2}^{n} \mathcal{E}(k) \left[A(\boldsymbol{U^{n-k+1}} - \boldsymbol{U^{n-k}}) + \boldsymbol{C^{n-k+1}} - \boldsymbol{C^{n-k}} \right] \\ &+ \mathcal{E}(1)(\boldsymbol{C^n} - \boldsymbol{C^{n-1}}) + \Delta t \boldsymbol{f^n}, \end{aligned} \tag{22}$$

where $\mathcal{E}(k) := \exp\left(-\frac{\alpha}{1-\alpha} k \Delta t\right)$. The vectors $\boldsymbol{U^n}$, $\boldsymbol{f^n}$ and $\boldsymbol{C^n}$ are defined just like in Section 4.1 while the matrix

$$B := \begin{pmatrix} -2s & s & 0 & \dots & 0 \\ s & -2s & s & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & s & -2s & s \\ 0 & \dots & 0 & s & -2s \end{pmatrix},$$

being $s := \frac{\mathcal{K}_\alpha}{\alpha \Delta x^2} \left[\exp\left(\frac{\alpha}{1-\alpha} \Delta t\right) - 1 \right]$.

Figures 1 to 3 exhibit the dependence of the computed values with respect to the nodes for the three first levels as regards the backward Euler type method when applied both to Caputo-Fabrizio as well as to Riemann-Liouville. The stencil composed of the red edges shows the values needed to be available in order to perform the calculations to be done at node i at step n. This visualization stresses the memory effect when dealing with the fractional derivative.

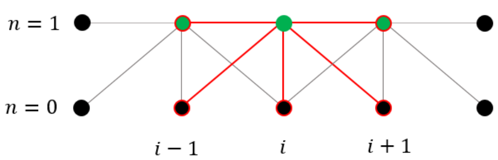


Figure 1: The stencil represents in red the approximation scheme at level $n = 1$ for Riemann-Liouville and Caputo-Fabrizio operators.

4.3 Katugampola fractional diffusion equation

The characteristics of Katugampola local operator let a numerical scheme to be drawn for the fractional derivative which makes use of only backward differences. This is allowed by the fact that

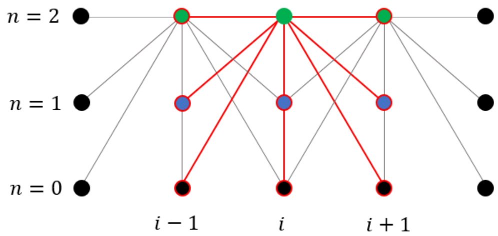


Figure 2: The stencil represents in red the approximation scheme at level $n = 2$ for Riemann-Liouville and Caputo-Fabrizio operators.

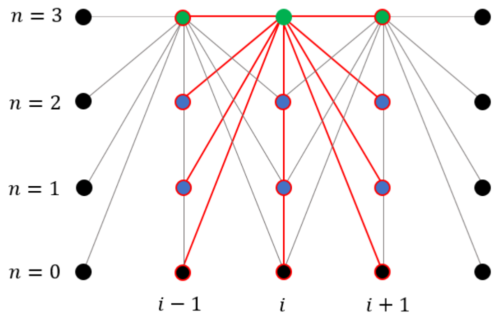


Figure 3: The stencil represents in red the approximation scheme at level $n = 3$ for Riemann-Liouville and Caputo-Fabrizio operators.

a formal relation between Katugampola derivative (6) and the order n ordinary derivative thanks to equation (7). For $0 < \alpha < 1$, we have $n = 0$, which implies

$$_K D^{\alpha} f(x) = t^{1-\alpha} f'(t).$$

Through replacement of the Riemann-Liouville fractional derivative by Katugampola in (9) plus application of the approximations (10) and (11), we are lead to

$$\frac{U_i^n - U_i^{n-1}}{\Delta t} = \mathcal{K}_{\alpha\, K} D^{\alpha} \left(\frac{U_{i-1}^n - 2U_i^n + U_{i+1}^n}{\Delta x^2} \right) + f(x_i, t_n). \tag{23}$$

We then apply backward differences to $\partial_x^2 U_i^n$ in (23) and multiply the result by $t^{1-\alpha}$, so as to reach the fractional derivative, for $0 < \alpha < 1$, i.e.,

$$\frac{U_i^n - U_i^{n-1}}{\Delta t} = \mathcal{K}_{\alpha} t_n^{1-\alpha} \left(\frac{\partial_x^2 U_i^n - \partial_x^2 U_i^{n-1}}{\Delta t} \right) + f(x_i, t_n). \tag{24}$$

This relation may be algebraically handled, for $1 \leq i \leq M - 1$, so as to lead to the matrix form of (24), namely

$$\boldsymbol{U^n} = \boldsymbol{U^{n-1}} + \mathcal{T}(n) \left(A(\boldsymbol{U^n} - \boldsymbol{U^{n-1}}) + \boldsymbol{C^n} - \boldsymbol{C^{n-1}} \right) + \Delta t \boldsymbol{f^n} \tag{25}$$

or

$$(\boldsymbol{I} - \mathcal{T}(n)A)\, \boldsymbol{U^n} = (\boldsymbol{I} - \mathcal{T}(n)A)\, \boldsymbol{U^{n-1}} + \mathcal{T}(n) \left(\boldsymbol{C^n} - \boldsymbol{C^{n-1}} \right) + \Delta t \boldsymbol{f^n} \tag{26}$$

where $\mathcal{T}(n) := (n)^{1-\alpha}$. The matrix A and the vectors $\boldsymbol{U^n}$, $\boldsymbol{f^n}$ and $\boldsymbol{C^n}$ are defined as in Section 4.1.

Figures 4 to 6 make available information for the backward Euler type method for Katugampola operator. This visualization confirms that Katugampola fracional derivative is a local operator just like the standard derivative.

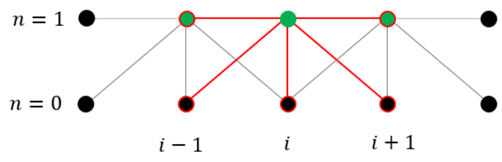


Figure 4: The stencil shows in red the approximation scheme at level $n = 1$ for Katugampola operator.

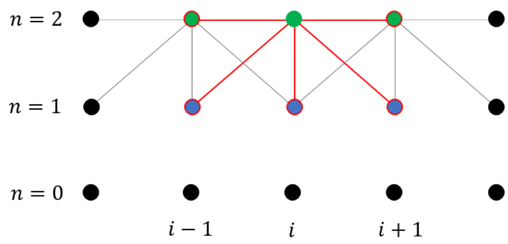


Figure 5: The stencil shows in red the approximation scheme at level $n = 2$ for Katugampola operator.

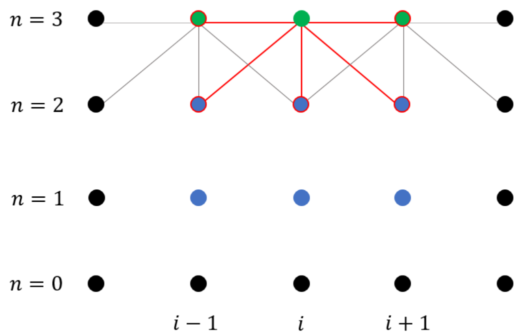


Figure 6: The stencil shows in red the approximation scheme at level $n = 3$ for Katugampola operator.

5 Numerical Results

In this section, some comparison tests are presented to show how the generated approximate solutions numerically compare with the exact solution, this one is known from an analytic expression obtained through the methods proposed in Section 4. These tests have as scope to evaluate the effect the dimensional correction brings in for the fractional diffusion equation for different values of α $(0 < \alpha < 1)$, for all three operators herein under study.

5.1 Convergence Order Study

The three tests exhibited in this subsection have as aim to present the convergence order with respect to the time variable for the three operators we are dealing with, being $0 < \alpha < 1$ and $\mathcal{R}_\alpha = \tau^\alpha$

taken relatively to the domain $(x,t) \in [0,1] \times [0,T]$. The chosen parameters were fixed as $\tau = 1$, $T = 1$, $\Delta x = 1/1000$, while different choices were made for the time variable, number of steps (N), as well as the order of the fractional derivative (α).

Table 1: Time variable order as regards to the numerical solution via backward Euler type method considering Riemann-Liouville operator.

N	$\alpha = 0.2$	order	$\alpha = 0.5$	order	$\alpha = 0.8$	order
8	3.2701×10^{-2}		4.0344×10^{-2}		2.5335×10^{-2}	
16	1.6365×10^{-2}	0.9987	2.0271×10^{-2}	0.9929	1.2959×10^{-2}	0.9672
32	8.1665×10^{-3}	1.0008	1.0148×10^{-2}	0.9956	6.5703×10^{-3}	0.9736
64	4.0723×10^{-3}	1.0018	5.0717×10^{-3}	0.9973	3.3127×10^{-3}	0.9784
128	2.0305×10^{-3}	1.0024	2.5333×10^{-3}	0.9983	1.6645×10^{-3}	0.9820

Table 2: Time variable order as regards to the numerical solution via backward Euler type method considering Caputo-Fabrizio operator.

N	$\alpha = 0.2$	order	$\alpha = 0,5$	order	$\alpha = 0,8$	order
10	8.8598×10^{-3}		1.6828×10^{-2}		2.2789×10^{-2}	
20	4.5167×10^{-3}	0.9720	8.4740×10^{-3}	0.9897	1.0023×10^{-2}	1.1850
40	2.2805×10^{-3}	0.9790	4.2545×10^{-3}	0.9919	4.6806×10^{-3}	1.1418
80	1.1459×10^{-3}	0.9836	2.1326×10^{-3}	0.9934	2.2640×10^{-3}	1.1105
160	5.7435×10^{-4}	0.9868	1.0683×10^{-3}	0.9944	1.1187×10^{-3}	1.0871

Table 3: Time variable order as regards to the numerical solution via backward Euler type method considering Katugampola operator.

N	$\alpha = 0.2$	order	$\alpha = 0.5$	order	$\alpha = 0.8$	order
50	2.4007×10^{-3}		9.3387×10^{-3}		2.2298×10^{-2}	
100	1.2113×10^{-3}	0.9869	4.6757×10^{-3}	0.9980	1.1156×10^{-2}	0.9991
200	6.0865×10^{-4}	0.9899	2.3427×10^{-3}	0.9975	5.6022×10^{-3}	0.9964
400	3.0514×10^{-4}	0.9920	1.1755×10^{-3}	0.9966	2.8230×10^{-3}	0.9939
800	1.5281×10^{-4}	0.9934	5.9154×10^{-4}	0.9952	1.4448×10^{-3}	0.9870

Table 1 presents the results for problem (9), taking into the function $u(x,t) = t^{1.5}\exp(\pi(x-\alpha))$ is the exact solution. See the corresponding data below.

- $f(x,t) = \left(1,5t^{0.5} - \pi^2 \frac{\Gamma(2.5)}{\Gamma(2.5-\alpha)} t^{1.5-\alpha}\right)\exp(\pi(x-\alpha));$
- $\phi(x) = 0; \quad l(t) = t^{1.5}\exp(-\pi\alpha); \quad r(t) = t^{1,5}\exp(\pi(1-\alpha)).$

Results presented in Table 2 refer to problem (9) with the Caputo-Fabrizio operator. The function $u(x,t) = t\exp(t)\sin(\alpha\pi x)$ is the exact solution, for data that follow:

- $f(x,t) = \left[((\pi^2\alpha^2+1)t + \pi^2\alpha^3 + 1)\exp(t) - \pi^2\alpha^3\exp(\frac{\alpha t}{\alpha-1})\right]\sin(\alpha\pi x);$
- $\phi(x) = 0; \quad l(t) = 0; \quad r(t) = t\exp(t)\sin(\alpha\pi).$

Just like in both previous tests, the results presented in Table 3, correspond to problem (9) for Katugampola operator. The function $u(x,t) = t\exp(t)\sin(\alpha\pi x)$ is the exact solution, and the associated data being listed in the sequel:

- $f(x,t) = (1 + t^{1-\alpha}\alpha^2\pi^2)(1+t)\exp(t)\sin(\alpha\pi x)$;
- $\phi(x) = 0$; $l(t) = 0$; $r(t) = t\exp(t)\sin(\alpha\pi)$.

Results shown on Tables 1, 2 and 3 may be considered as very consistent, based on verification of the expected order as calculated with the norm L^2.

5.2 Influence of order α and parameter τ

Tests in this subsection present the efects lead by changes on the fractional derivative order α and the dimensional correction parameter τ in the diffusion equation for the chosen fractional operators.

The approximate solution presented on Figures from 7 to 9 refer to our diffusion problem, with $(0 < \alpha < 1)$ e $\mathcal{R}_\alpha = \tau^\alpha$ throughout the domain $(x,t) \in [0,1] \times [0,2]$, with related data in sequel:

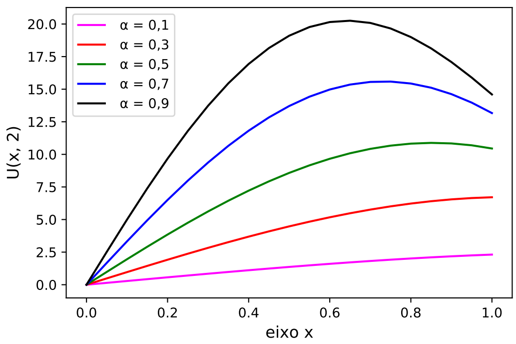
- $f(x,t) = \left[((\pi^2\alpha^2 + 1)t + \pi^2\alpha^3 + 1)\exp(t) - \pi^2\alpha^3\exp(\frac{\alpha t}{\alpha - 1})\right]\sin(\alpha\pi x)$;
- $\phi(x) = 0$; $l(t) = 0$; $r(t) = t\exp(t)\sin(0.5\alpha\pi)$,

taking $\Delta x = 1/20$ e $\Delta t = 2/200$. Figures 7(a)-(c) show the effect caused by changes in the order α for the fractional derivative associated to Riemann-Liouville, Caputo-Fabrizio and Katugampola operators, respectively, having the parameter τ value being fixed as one.

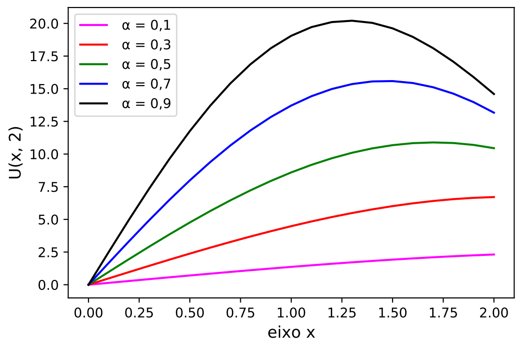
Figures from 8(a) to 8(f) show surfaces drawn as graphs where the values for α were chosen as 0.1 and 0.9. Riemann-Liouville operator option appears in Figures 8(a) and 8(b), while Figures 8(c) and 8(d) show Caputo-Fabrizio and finally Katugampola data belong to Figures 8(e) and 8(f). For the parameter τ, its value was kept as one.

Figures 9(a)-(c) exhibit numerical results for different values of the dimensional correction parameter τ and the fracional derivative order α. Figure 9(a) gives the results for Riemann-Liouville operator, for Caputo-Fabrizio Figure 9(b) was chosen and we finish with Katugampola in Figure 9(c).

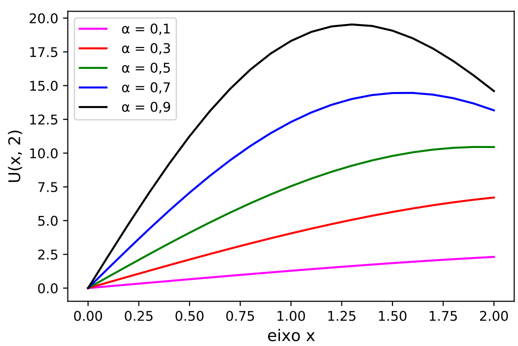
Figure 10 presents a numerical solution when considering all three chosen fractional operatos, with the value of α fixed as 0.5 and the parameter $\tau = 2$.



(a) Riemann-Liouville operator

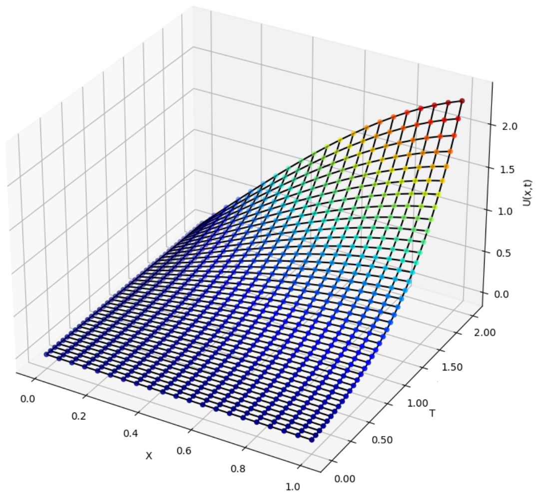


(b) Caputo-Fabrizio operator

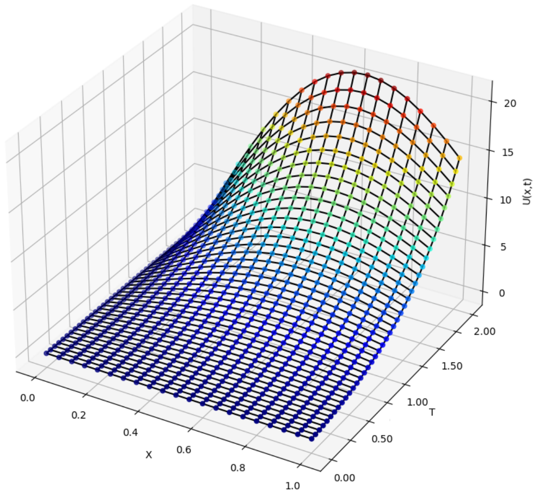


(c) Katugampola operator

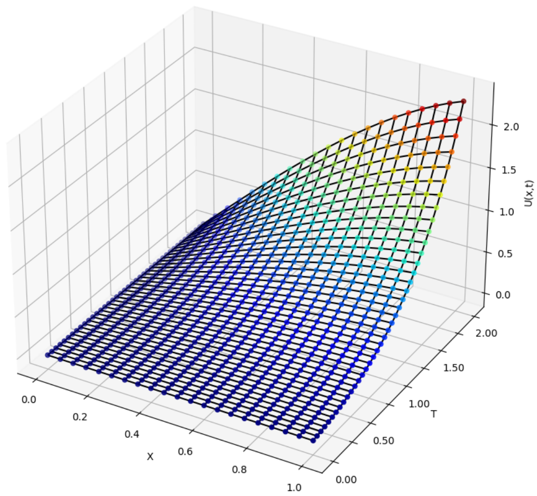
Figure 7: The efect of α in the numerical solution associated to (a) Riemann-Liouville, (b) Caputo-Fabrizio and (c) Katugampola operators having $\tau = 1.0$.



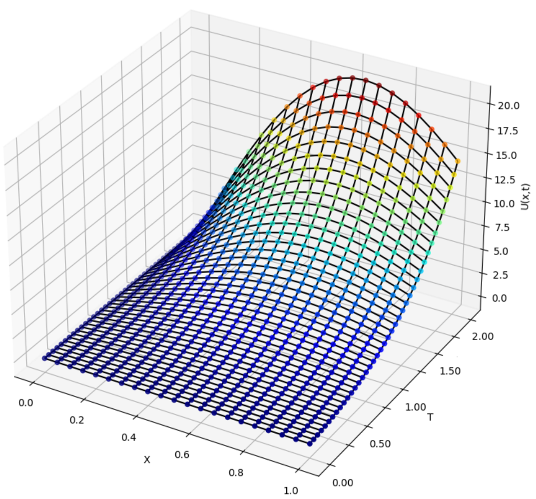
(a) $\alpha = 0.1$



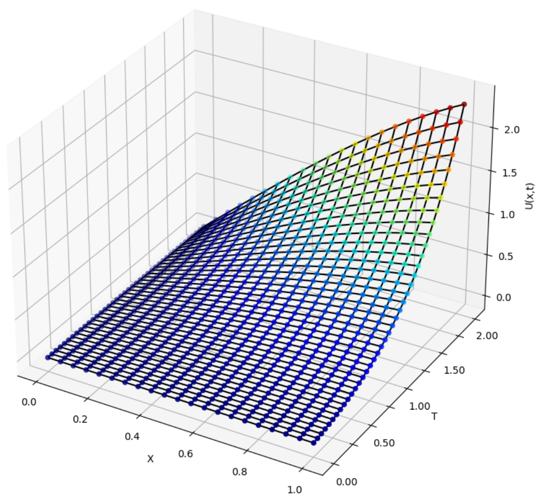
(b) $\alpha = 0.9$



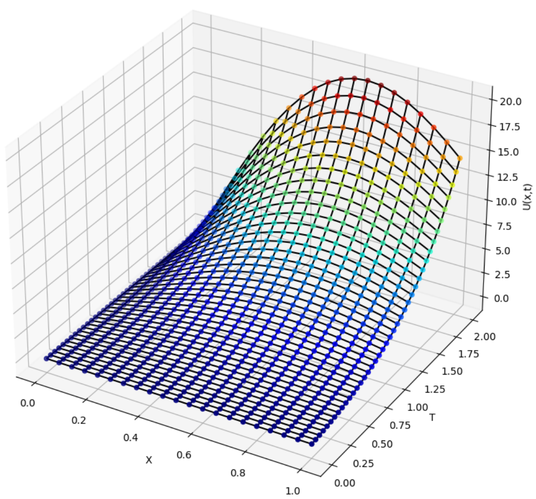
(c) $\alpha = 0.1$



(d) $\alpha = 0.9$

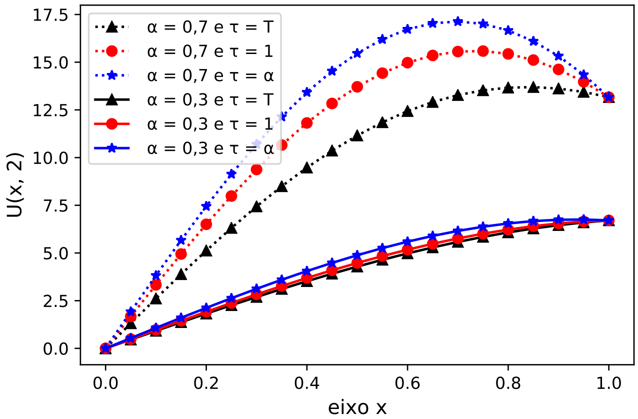


(e) $\alpha = 0.1$

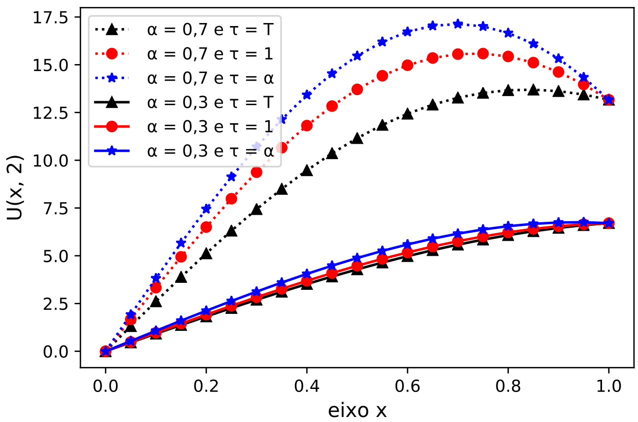


(f) $\alpha = 0.9$

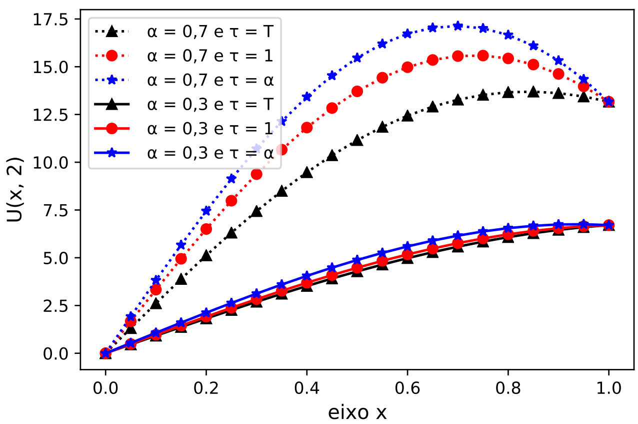
Figure 8: Numerical solution associated to (a)-(b) Riemann-Liouville, (c)-(d) Caputo-Fabrizio and (e)-(f) Katugampola operators.



(a) Riemann-Liouville operator



(b) Caputo-Fabrizio operator



(c) Katugampola operator

Figure 9: Numerical solution associated to (a) Riemann-Liouville, (b) Caputo-Fabrizio and (c) Katugampola operators for $T = 2.0$.

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DOI: 10.21167/cqdv22n22022102118 Disponível em: www.fc.unesp.br/departamentos/matematica/revista-cqd

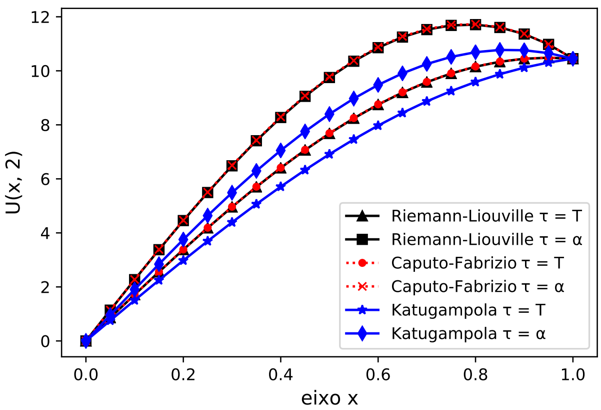

Figure 10: Numerical solutions at $T = 2$ with $\alpha = 0.5$.

6 Conclusions

The computational tests performed for this work have suggested that the backward Euler type numerical method is unconditionally stable for the three operators we have considered. All our computer runs have exhibited results quite dependable, as well as compatible with those obtained while dealing with its classical formulation for the integer order derivatives diffusion equation, and which has given our inspiration to suggest the hereby schemes.

Moreover, our numerical results indicate that the fractional derivative order α points to a strong influence in the modeling, as already expected. Nevertheless, insertion of the dimension correction parameter τ in the modeling would force us to deal with a chalenge, namely, an optimization problem to be even formulated (and solved): how to choose the best value for τ. This certainly amounts to an important research track, as a theoretical suport is also carried over: to bypass dimensional errors in the formulation.

The fractional operators dealt with belong to different classes according to (TEODORO, 2019) classification, which is based in the criterion proposed by (ORTIGUEIRA; MACHADO, 2015). Despite of that, the numerical results generated by employing both Riemann-Liouville and Caputo-Fabrizio operators have remained quite close, independently of the choice for α and τ, as opposite to Katugampola and its slightly different results.

Our outgoing research plans cover the theoretical analysis of stability for the employed algorithms as well as the search for alternate approximation schemes for fractional derivatives looking up for better precision. Besides, we look forward to testing the model with data obtained from real life environments and test them with respect to better choices for the dimension values.

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