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The e-positive mild solutions for fractional impulsive evolution equations with $1 < \alpha < 2$

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Abstract

In this paper, we investigate the existence of global e-positive mild solutions for impulsive evolution fractional differential equation in the Caputo sense of order $1 < \alpha < 2$. To obtain the result, we usually the theory of sectorial operators, theory of Kuratowski measure of noncompactness, Cauchy criterion and the Gronwall inequality.

Keywords: Fractional Differential Equation. e-positive mild solutions. Kuratowski measure of noncompactness. Gronwall inequality. α-resolvent family.

1 Introduction

In the present paper, we consider the system with nonlinear impulsive evolution fractional differential equation, given by

$$\begin{cases} {}^{C}\mathbb{D}^{\alpha}_{0+}u(t) + Au(t) = f(t,\ u(t)), & t \in J_\infty = [0,+\infty),\ \ \alpha \in (1,2), \\ \Delta u(t_k) = I(u(t_k)), \quad \Delta u'(t_k) = I(u'(t_k)), & k \in \mathbb{N},\ \ t \neq t_k, \\ u(0) = x_0, \qquad u'(0) = x_1, \end{cases} \tag{P}$$

where ${}^{C}\mathbb{D}^{\alpha}_{0+}(\cdot)$ is the Caputo fractional derivative of order $\alpha \in (1,2)$, $u : J_\infty \longrightarrow E$, $J_\infty = [0,+\infty)$ e $(E, \|\cdot\|)$ is a Banach space. The operator $A : D(A) \subset E \rightarrow E$ is sectorial of type (M,θ,α,μ), $f \in C(J_\infty \times E, E)$, $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$ is the jump of the function u at the point t_k that represents the impulse function $I : E \rightarrow E$, where $u(t_k^+)$ and $u(t_k^-)$ represent the limits on the right and on the left sides of the $u(t)$ in $t = t_k$, respectively; analogue to $\Delta u'|_{t=t_k}$ e $x_0, x_1 \in E$. Also consider, $0 < t_1 < t_2 < \cdots < t_m \cdots$, with $t_m \xrightarrow{m\to\infty} \infty$, a partition on J_∞ and define the ranges: $J'_\infty = J_\infty \backslash \{t_1, t_2, \ldots, t_m, \ldots\}$, $J_0 = [0, t_1]$ and $J_k = (t_k, t_{k+1}]$ $(k \in \mathbb{N})$. Also, let λ_1 be the smallest positive real eigenvalue of the linear operator A and $e_1 \in D(A)$ the positive eigenvector corresponding to λ_1.

Currently, fractional calculus is well solidified with numerous definitions of fractional derivatives and fractional integrals. These fundamental concepts of fractional calculus are being used in many other areas, producing numerous results (SOUSA; OLIVEIRA, 2018a, 2019a, 2019b; SOUSA; FREDERICO; OLIVEIRA, 2020). An important consequence that the fractional calculus provides, is the investigation of properties regarding fractional differential equations, the main point of this work. There are already a large number of works published in this area of fractional differential equations, as it has aroused interest in the scientific community (CHEN; ZHANG; LI, 2019; EIDELMAN; KOCHUBEI, 2004; SOUSA; JARAD; ABDELJAWAD, 2021). Researchers justify that working with fractional operators (derivatives and integrals) produce better results when compared to classical operators (full order), in particular, when it comes to applications (NIKAN; AVAZZADEH; MACHADO, 2020; SOUSA; OLIVEIRA; MAGNA, 2017; SOUSA, 2018). There is still a vast field to be explored, as the theory has been built in different directions of the fractional differential equation theory, involving sectorial and quasi-sectorial operators (SOUSA; OLIVEIRA; MAGNA, 2017; SOUSA, 2018; WANG; FECKAN; ZHOU, 2013; WANG; SHU, 2015). Furthermore, there are still many questions, which when answered will enrich the theory. Thus, we highlight some relevant works involving sectorial and quasi-sectorial operators (CHEN; ZHANG; LI, 2020; DING; AHMAD, 2016; WANG; CHEN; XIAO, 2012; YANG; LIANG, 2013; ZHANG; LIANG, 2018).

In 2012, Shu and Wang (SHU; WANG, 2012) investigated the existence and uniqueness of a mild solution for a system with a semilinear fractional integrodifferential equation in a Banach space, using the Krasnoselskii theorem, the Arzelà-Ascoli theorem and the theorem of fixed point. In 2013, Yang and Liang (YANG; LIANG, 2013) investigated positive solutions to the Cauchy problem of evolution fractional equations via Caputo fractional derivative in Banach spaces, using fixed-point theorems and semigroup analytic theory. Still in 2013, Wang et al.(WANG; FECKAN; ZHOU, 2013) investigated the existence of piecewise continuous mild solutions and applications of fractional impulsive parabolic control in a study on optimal control for nonlinear impulsive evolutionary fractional equations.

In 2015, Wang and Shu (WANG; SHU, 2015) investigated the existence of positive mild solutions of fractional evolution equations with nonlocal conditions of order $1 < \alpha < 2$, using Schauder fixed point theorem and the theorem of Krasnoselskii fixed point. See also work by Ding and Ahmad (DING; AHMAD, 2016). See to (SHAH et al., 2018; CHEN; LI, 2010; CHEN; ZHANG; LI, 2020; RAMOS; SOUSA; OLIVEIRA, 2022; SHU; WANG, 2012; SOUSA; BENCHOHRA; N'GUÉRÉKATA, 2020; SOUSA; OLIVEIRA, 2018b) and the references therein.

Motivated by the works discussed above and with the purpose of providing new results in order to significantly contribute to the area of fractional differential equations, we will now present in detail the main objectives obtained in this paper. So we have:

1. First, we present a new class of fractional differential equations with impulses and order $1 < \alpha \leq 2$, in addition to the respective class of mild solutions via resolvent operators.

2. We discuss the necessary and sufficient conditions for the existence of e-positive mild solutions, and we resort to the theory of Kuratowski measure of noncompactness, Cauchy criterion and Gronwall inequality.

The rest of the article is divided into: Section 2, which presents some essential concepts and results for the discussion of the main result. In Section 3, we investigate the existence of e-positive mild solutions via Kuratowski measure of noncompactness and Gronwall inequality.

2 Preliminaries

We present some fundamental concepts and results that will be useful.

Consider the Banach space $(E, \|\cdot\|)$, the interval $J = [a, b] \subset \mathbb{R}$ and $n \in \mathbb{N}$. The space of continuous functions and its usual norm are given, respectively by

$$C(J, E) := \{f : J \to E;\ f : \text{continuous}\} \quad \text{and} \quad \|f\|_C := \sup_{t\in J} |f(t)|.$$

The space of the continuously differentiable n-times functions and its usual norm are given, respectively, by

$$C^n(J, E) := \left\{f : J \to E;\ f^{(n)} \in C(J, E)\right\} \quad \text{and} \quad \|f\|_{C^n} := \sup_{t\in J} |f^{(n)}(t)|.$$

Note that the spaces defined above are Banach spaces. Now consider, the real interval $J_\infty = [0, \infty)$. The space of continuous functions by parts given by

$$PC(J_\infty, E) := \left\{\begin{matrix} u : J_\infty \to E;\ u(t) : \text{continuous in } t \neq t_k,\ \text{continuous left} \\ \text{in } \ t = t_k \text{ and there is the limit on the right}, u(t_k^+),\ \forall k \in \mathbb{N} \end{matrix}\right\},$$

equipped with the norm $\|u\|_{PC} = \sup\{\|u(t)\|;\ t \in J\}$ is a Banach space.

Definition 2.1 (Cone) *Let E be a real Banach space. A non-empty, closed, and convex subset $E^+ \subset E$ is said to be a cone if it satisfies the following conditions:*

(i) *If $x \in E^+$ and $\lambda \geqslant 0$, then $\lambda x \in E^+$.*

(ii) *If $x \in E^+$ and $-x \in E^+$, then $x = 0$.*

Every cone $E^+ \subset E$ induces an order in E given by: $x \leqslant y \Leftrightarrow y - x \in E^+$.

Let $J = [a, b] \subset \mathbb{R}$ be an interval with $-\infty \leqslant a < b \leqslant \infty$. The Riemann-Liouville fractional integral on the left side of a function f in J of order $\alpha > 0$ is defined by

$$\mathcal{I}_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a. \tag{1}$$

Analogously, we define the corresponding version on the right side.

On the other hand, let $n \in \mathbb{N}$ and $J = [a,\ b] \subset \mathbb{R}$ be an interval such that $-\infty \leqslant a < b \leqslant \infty$. Also consider the functions $f \in C^n_{\alpha,\beta}(J;\ \mathbb{R})$. The Caputo fractional derivative on the left side of f of order $\alpha \in (n-1, n)$ and type $\beta \in [0,1]$ is defined as (KILBAS; SRIVASTAVA; TRUJILLO, 2006; SOUSA; OLIVEIRA, 2018a)

$${}^{C}\mathbb{D}_{a+}^{\alpha} f(x) = \mathcal{I}_{a+}^{n-\alpha} \left(\frac{d}{dx}\right)^n f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t) dt. \tag{2}$$

Analogously, we define the corresponding version on the right.

For details on how to obtain other particular cases for derivatives and fractional integrals, we suggest the work (SOUSA; OLIVEIRA, 2018a). Next, we will present two fundamental results, the Theorem 2.2 and the Lemma 2.3. The proof can be found in the paper (SOUSA; OLIVEIRA, 2019).

Theorem 2.2 (SOUSA; OLIVEIRA, 2019) *Let u and v be two integrable functions and g continuous, with domain $J = [a,b]$. Let $\psi \in C^1(J)$ be an increasing function such that $\psi'(t) \neq 0$, $\forall t \in J$. suppose that*

(1) *u and v are non-negative;*

(2) *g is non-negative and non-descending.*

If

$$u(t) \leqslant v(t) + g(t) \int_a^t \psi'(\tau)\Big(\psi(t) - \psi(\tau)\Big)^{\alpha-1} u(\tau) d\tau, \tag{3}$$

then

$$u(t) \leqslant v(t) + \int_a^t \sum_{k=1}^{\infty} \frac{[g(t)\Gamma(\alpha)]^k}{\Gamma(\alpha k)} \psi'(\tau)\Big(\psi(t) - \psi(\tau)\Big)^{\alpha k-1} v(\tau) d\tau. \tag{4}$$

Lemma 2.3 (SOUSA; OLIVEIRA, 2019) *Under the hypothesis of* Theorem 2.2*, let v be a non-descending function on $J = [a,b]$. So we have*

$$u(t) \leqslant v(t) \mathbb{E}_{\alpha} \left(g(t)\Gamma(\alpha)\Big[\psi(t) - \psi(a)\Big]^{\alpha}\right), \quad \forall\, t \in J,$$

where $\mathbb{E}_{\alpha}(t) = \sum\limits_{k=0}^{\infty} \dfrac{t^k}{\Gamma(\alpha k + 1)}$, with $\Re(\alpha) > 0$, is the one-parameter Mittag-Leffler function.

Definition 2.4 (SHU; WANG, 2012; SHU; SHI, 2016) *Let A be a densely defined closed linear operator on a Banach space E. The bounded set $\{S_\alpha(t);\ t \geqslant 0\}$ is considered a α-resolvent family generated by A if the following conditions are satisfied:*

(*a*) *$S_\alpha(\cdot)$ is strongly continuous in $\mathbb{R}_+$ and $S_\alpha(0) = I$;*

(*b*) $S_\alpha(t)D(A) \subset D(A)$ *and* $AS_\alpha(t)x = S_\alpha(t)Ax$ *for all* $x \in D(A)$*,* $t \geqslant 0$*;*

(*c*) *For all* $x \in D(A)$ *and* $t \geqslant 0$*,* $S_\alpha(t)x = x + \mathcal{I}_0^\alpha S_\alpha(t)Ax$*.*

Definition 2.5 (SHU; WANG, 2012; SHU; SHI, 2016) *A closed linear operator* $A : D(A) \subset X \to X$ *is considered a sectorial operator of type* (M, θ, α, μ) *if there is* $0 < \theta < \frac{\pi}{2}$*,* $M > 0$ *and* $\mu \in \mathbb{R}$ *such as*

1. *the* α*-resolvent of* A *exists outside the sector* $\mu + S_\theta = \{\mu + \lambda^\alpha;\ \lambda \in \mathbb{C}, |\arg(-\lambda^\alpha)| < \theta\}$*,*
2. *and, satisfy the estimate* $\|(\lambda^\alpha I - A)^{-1}\| \leqslant \dfrac{M}{|\lambda^\alpha - \mu|}$*,* $\quad \lambda^\alpha \notin \mu + S_\theta$*.*

If A is a sectorial operator of type (M, θ, α, μ), then it is not difficult to see that A is the infinitesimal generator of a α-resolvent family of the solution operators $S_\alpha(\cdot)$, $T_\alpha(\cdot)$ e $K_\alpha(\cdot)$, in a Banach space, (SOUSA; BENCHOHRA; N'GUÉRÉKATA, 2020; SOUSA; OLIVEIRA; MAGNA, 2017).

In order to obtain the existence of a e-positive mild solution of the system (P), we present the concept of Kuratowski measure of noncompactness and some important results of it.

Definition 2.6 (Kuratowski measure of noncompactness) (WANG; ZHOU; FECKAN, 2013) *Let* B *be a bounded set in a Banach space* E *and let* $\delta(X)$ *be the diameter of a set* X*. The Kuratowski measure of noncompactness* $\mu(\cdot)$ *is given by,*

$$\mu(B) = \inf\left\{\varepsilon > 0;\ B = \bigcup_{i=1}^{m} B_i \ e\ \delta(B_i) \leqslant \varepsilon,\ \forall i \in [1..m]\right\}. \tag{5}$$

The Kuratowski measure of noncompactness guarantees that every bounded set B admits finite coverage, i.e., B can be covered by a finite number of sets with a diameter not greater than $\varepsilon > 0$.

Lemma 2.7 (WANG; ZHOU; FECKAN, 2013) *Let* S *and* T *be bounded sets in a Banach space* E*, let* $\bar{S}$ *be the closing of* S*,* $\overline{co}(S)$ *the convex hull of* S *and* a *a real number. So the measure of noncompactness has the following properties*

(1) $\mu(S) = 0 \iff \overline{S}$ *is compact;*

(2) $S \subset T \quad \Rightarrow \quad \mu(S) \leqslant \mu(T)$*;*

(3) $\mu(S) = \mu(\overline{S}) = \mu(\overline{co}(S))$*;*

(4) $\mu(aS) = |a|\ \mu(S)$*;*

(5) $\mu(S + T) \leqslant \mu(S) + \mu(T)$*, where* $S + T = \{x + y;\ x \in S, y \in T\}$*;*

(6) $\mu(S \cup T) = \max\{\mu(S),\ \mu(T)\}$*;*

(7) $\mu(\{x\} \cup S) = \mu(S), \quad \forall x \in E, \quad \emptyset \neq S \subset E$*;*

For the next lemmas, consider the interval $J = [0, b]$ and the Banach space $C(J, E)$, then for any $B \subset C(J, E)$ and for all $t \in J$, we define the sets

$$B(t) := \{u(t);\ u \in B\} \subset E \qquad \text{and} \qquad \int_0^t B(s)ds := \left\{\int_0^t u(s)ds;\ u \in B\right\}.$$

Lemma 2.8 (WANG; ZHOU; FECKAN, 2013) *Let $B \subset C(J;E)$ be a bounded set, then $B(t)$ is bounded in E and*

$$\mu(B(t)) \leqslant \mu(B), \text{ for all } t \in J.$$

Lemma 2.9 (WANG; ZHOU; FECKAN, 2013) *Let $B \subset C(J,E)$ be bounded and equicontinuous, then $\mu(B(t))$ is continuous in J,*

$$\mu(B) = \sup\left\{\mu(B(t));\ t \in J\right\} \qquad \text{and} \qquad \mu\left(\int_0^t B(s)ds\right) \leqslant \int_0^t \mu\Big(B(s)\Big)ds.$$

Lemma 2.10 (WANG; ZHOU; FECKAN, 2013) *Let $J = [a,b]$, $B \subset C(J;\ E)$ be bounded and equicontinuous, then $\overline{co}(B) \subset C(J;\ E)$ is also bounded and equicontinuous.*

Lemma 2.11 (WANG; ZHOU; FECKAN, 2013) *Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of Bochner-integrable functions, from $J = [a,b]$ in E, with $\|u_n(t)\| \leqslant \overline{m}(t)$, for almost every $t \in J$ and every $n \geqslant 1$, where $\overline{m} \in L(J;\mathbb{R}_+)$, then the function $\Phi(t) = \mu\Big(\{u_n(t)\}_{n=1}^{\infty}\Big) \in L(J;\mathbb{R}_+)$ and satisfies*

$$\mu\left(\left\{\int_a^t u_n(s)ds;\ n \in \mathbb{N}\right\}\right) \leqslant 2\int_a^t \Phi(s)ds. \tag{6}$$

Lemma 2.12 (WANG; ZHOU; FECKAN, 2013) *Let B be bounded, then for every $\varepsilon > 0$, there is a sequence $\{u_n\}_{n=1}^{\infty} \subset B$, such that*

$$\mu(B) \leqslant \mu\Big(\{u_n\}_{n=1}^{\infty}\Big) + \varepsilon.$$

3 Existence of e-positive mild solution

We investigate the existence of e-positive mild solutions for an initial value problem with nonlinear impulsive evolution fractional differential equation in a Banach space, through the Gronwall inequality, Cauchy criterion and the noncompactness measure by Kuratowski (SOUSA; OLIVEIRA, 2019; WANG; ZHOU; FECKAN, 2013).

Definition 3.1 (Mild solution) (SHU; WANG, 2012; SHU; SHU; XU, 2019; SHU; SHI, 2016) *An abstract function $u \in PC(J_\infty, E)$ is a mild solution to the* system (P) *if it satisfies the following integral equation*

$$u(t) = S_\alpha(t)x_0 + K_\alpha(t)x_1 + \int_0^t T_\alpha(t-s)f(s,\ u(s))ds + S_\alpha(t)\sum_{i=1}^{k} S_\alpha^{-1}(t_i)I(u(t_i)) + K_\alpha(t)\sum_{i=1}^{k} K_\alpha^{-1}(t_i)I(u'(t_i)),$$

with $S_\alpha(\cdot)$, $K_\alpha(\cdot)$ and $T_\alpha(\cdot)$ given by

$$S_\alpha(t) = \mathbb{E}_{\alpha,1}(At^\alpha) = \sum_{k=0}^{\infty}\frac{(At^\alpha)^k}{\Gamma(1+\alpha k)}, \quad T_\alpha(t) = t^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(At^\alpha) = t^{\alpha-1}\sum_{k=0}^{\infty}\frac{(At^\alpha)^k}{\Gamma(\alpha+\alpha k)}.$$

and

$$K_\alpha(t) = t\mathbb{E}_{\alpha,2}(At^\alpha) = t\sum_{k=0}^{\infty}\frac{(At^\alpha)^k}{\Gamma(2+\alpha k)}.$$

Remembering that, $S_\alpha^{-1}(\cdot)$ and $K_\alpha^{-1}(\cdot)$ denote the inverse of the solution operators $S_\alpha(\cdot)$ and $K_\alpha(\cdot)$, respectively, at $t = t_i$, $i = 1,2,3,\cdots,m$. Also, if there is $e \geqslant 0$ and $\sigma > 0$, so that $u(t) \geqslant \sigma e$ for $t \in J_\infty$, so we call it an e-positive mild solution of the system (P).

Let $(E, \|\cdot\|)$ be a Banach space, $A : D(A) \subset E \to E$ a closed linear operator and $-A$ the infinitesimal generator of the α-resolvent families $\{S(t);\ t \geqslant 0\}$, $\{K(t);\ t \geqslant 0\}$ and $\{T(t);\ t \geqslant 0\}$. So there are $\widetilde{M} > 0$ and $\delta > 0$ such that (WANG; ZHOU; FECKAN, 2013; WANG; CHEN; XIAO, 2012)

$$\|S_\alpha(t)\|_C \leqslant \widetilde{M}e^{\delta t}, \quad \|K_\alpha(t)\|_C \leqslant \widetilde{M}e^{\delta t} \qquad \text{and} \qquad \|T_\alpha(t)\|_C \leqslant \widetilde{M}e^{\delta t}, \quad t \geqslant 0.$$

Theorem 3.2 *Let $(E, \|\cdot\|)$ be a Banach space with partial order "$\leqslant$", whose positive cone E^+ is normal, and $-A$ is the infinitesimal generator of the positive α-resolvent families $\{S_\alpha(t);\ t \geqslant 0\}$, $\{K_\alpha(t);\ t \geqslant 0\}$ and $\{T_\alpha(t);\ t \geqslant 0\}$. For a constant $\sigma > 0$ and $t \in J_\infty$, let $x_0 \geqslant \sigma e_1$ and $f(t, \sigma e_1) \geqslant \lambda_1 \sigma e_1$. If the nonlinearity of $f \in C(J_\infty \times E^+, E)$ satisfies the following conditions:*

(H$_1$) *For $t \in J_\infty$ and $x \in E^+$, there are functions a and $b \in C(J_\infty, E^+)$, such that $\|f(t,x)\| \leqslant a(t)\|x\| + b(t)$.*

(H$_2$) *For every $R > 0$ and $T > 0$, there is $C = C(R,T) > 0$, such that $f(t,x_2) - f(t,x_1) \geqslant -C \cdot (x_2 - x_1)$, for all $t \in [0,T]$ and for $0 \leqslant x_1 \leqslant x_2$, with $\|x_1\|, \|x_2\| \leqslant R$.*

(H$_3$) *For every $R > 0$ and $T > 0$, there is $L = L(R,T) > 0$, such that every monotonous increasing sequence $D = \{x_n\} \subset E^+ \cap \overline{B}(0,R)$ satisfy $\mu\Big(f(t,D)\Big) \leqslant L\mu(D), \quad \forall\, t \in [0,T]$.*

So the system (P) *have e-positive mild solution in J_∞.*

Proof 3.3 ***(I)*** *Global existence of e-positive mild solutions in the interval $J_0 = [0, t_1]$. Note that the system* (P) *is equivalent to the system* (7) *with the evolution fractional equation without impulse in E,*

$$\begin{cases} {}^C\mathbb{D}^\alpha_{0+}u(t) + Au(t) = f(t, u(t)), & t \in J_0, \\ u(0) = x_0, \quad u'(0) = x_1. \end{cases} \tag{7}$$

Now, we subdivide this first part into two steps:

(A) *The local existence of mild solutions for the system* (7) *in $J_0 = [0, t_1]$.*

For every $t_0 \geqslant 0$ and $x_0, x_1 \in E$, the system (8) *with evolution fractional equation*

$$\begin{cases} {}^C\mathbb{D}^\alpha_{t_0+}u(t) + Au(t) = f(t, u(t)), & t > t_0, \\ u(t_0) = x_0, \quad u'(t_0) = x_1, \end{cases} \tag{8}$$

admits an e-positive mild solution em $I = [t_0, t_0 + h_{t_0}]$, where $h_{t_0} \in (0,1)$ will be defined later.

Consider the interval $I_* = [0, t_0 + 1]$ and write:

(i) *The constants*

$$M_{t_0} = \sup\left\{(t - t_0)^{2-\alpha}\|S_\alpha(t)\|;\ t \in I_*\right\} \text{ and } \widetilde{M}_{t_0} = \sup\left\{(t - t_0)^{2-\alpha}\|K_\alpha(t)\|;\ t \in I_*\right\}$$

$$\overline{M}_{t_0} = \sup\left\{(t - t_0)^{2-\alpha}\|T_\alpha(t)\|;\ t \in I_*\right\} \text{ and } R_{t_0} = (M_{t_0} + \overline{M}_{t_0})(\|x_0\| + 1) + \sigma e_1 + \widetilde{M}_{t_0}\|x_1\|.$$

(ii) *Let a and b be the functions in the condition* (H$_1$)*, then*

$$a_{t_0} = \max_{t \in I_*} a(t) \quad \text{and} \quad b_{t_0} = \max_{t \in I_*} b(t).$$

(*iii*) *Let the constants in the conditions* (H_2) *and* (H_3), *respectively,*

$$C = C(R_{t_0}, t_0 + 1) \quad \text{and} \quad L = L(R_{t_0}, t_0 + 1).$$

Adding the portion $Cu(t)$ *on both sides of the equation in the system* (8), *we can rewrite it as*

$$\begin{cases} {}^{C}\mathbb{D}^{\alpha}_{t_0+}u(t) + (A + CI)u(t) = f(t, u(t)) + Cu(t), & t > t_0, \\ u(t_0) = x_0, \quad u'(t_0) = x_1. \end{cases} \tag{9}$$

Consider the operators $\widetilde{S}_\alpha(t) = e^{-Ct}S_\alpha(t)$, $\widetilde{K}_\alpha(t) = e^{-Ct}K_\alpha(t)$ *and* $\widetilde{T}_\alpha(t) = e^{-Ct}T_\alpha(t)$ *belonging, respectively, to positive* α*-resolvent families* $\{S(t);\ t \geqslant 0\}$, $\{K(t);\ t \geqslant 0\}$ *and* $\{T(t));\ t \geqslant 0\}$, *all generated by* $-(A + CI)$. *Define the application* Λ *as*

$$(\Lambda u)(t) = \widetilde{S}_\alpha(t - t_0)x_0 + \widetilde{K}_\alpha(t - t_0)x_1 + \int_{t_0}^{t} \widetilde{T}_\alpha(t - s)\Big[f(s, u(s)) + Cu(s)\Big]ds, \quad t \in I. \tag{10}$$

Note that the function $\Lambda : C(I, E^+) \to C(I, E)$ *is continuous and increasing, because* f *is continuous and because of the condition* (H_2). *Furthermore, a fixed point of* Λ *is also a solution of the system* (9) *in* I.

Define the set Ω, *given by*

$$\Omega := \Big\{u \in C(I, E^+);\ \|u(t)\|_C \leqslant R_{t_0},\ u(t) \geqslant \sigma e_1,\ t \in I\Big\}.$$

Then, $\Omega \subset C(I, E^+)$ *is non-empty, bounded, convex and closed. Let* h_{t_0}, *such that*

$$(h_{t_0})^\alpha \leqslant \min\left\{1,\ \frac{(\|x_0\|+1)\alpha}{(a_{t_0} + C)R_{t_0} + b_{t_0}}\right\}.$$

Then by Eq.(10) *and by the condition* (H_1), *for each* $u \in \Omega$ *and* $t \in I$, *yields*

$$\begin{aligned} \|(\Lambda u)(t)\| &= \left\|\widetilde{S}_\alpha(t - t_0)x_0 + \widetilde{K}_\alpha(t - t_0)x_1 + \int_{t_0}^{t} \widetilde{T}_\alpha(t - s)\Big[f(s, u(s)) + Cu(s)\Big]ds\right\| \\ &\leqslant \|\widetilde{S}_\alpha(t - t_0)\|\|x_0\| + \|\widetilde{K}_\alpha(t - t_0)\|\|x_1\| + \int_{t_0}^{t} \left\|\widetilde{T}_\alpha(t - s)\right\| \|f(s, u(s)) + Cu(s)\|\, ds \\ &\leqslant M_{t_0}\|\|x_0\| + \widetilde{M}_{t_0}\|\|x_1\| + \overline{M}_{t_0}\Big[(a_{t_0} + C)R_{t_0} + b_{t_0}\Big]\int_{t_0}^{t} (t - s)^{\alpha-2}ds. \end{aligned}$$

Above, we used the fact that $(t - s)^{2-\alpha}\left\|\widetilde{T}_\alpha(t - s)\right\| \leqslant \overline{M}_{t_0}$, *and as* $(t - t_o) < h_{t_0}$, *follows*

$$\begin{aligned} \|(\Lambda u)(t)\| &\leqslant M_{t_0}\|x_0\| + \widetilde{M}_{t_0}\|\|x_1\| + \overline{M}_{t_0}\Big[(a_{t_0} + C)R_{t_0} + b_{t_0}\Big]\frac{(t - t_0)^\alpha}{\alpha} \\ &\leqslant M_{t_0}\|x_0\| + \widetilde{M}_{t_0}\|\|x_1\| + \overline{M}_{t_0}\frac{[(a_{t_0} + C)R_{t_0} + b_{t_0}]}{\alpha}\frac{(\|x_0\|+1)\alpha}{[(a_{t_0} + C)R_{t_0} + b_{t_0}]} \\ &\leqslant [M_{t_0} + \overline{M}_{t_0}]\,(\|x_0\|+1) + \widetilde{M}_{t_0}\|\|x_1\| \leqslant R_{t_0}. \end{aligned}$$

Now, let $v_0(t) = \sigma e_1$, $\forall t \in I$, *from this it follows that* $v_0 \in \Omega$. *Thus*

$$\varphi(t) := {}^{C}\mathbb{D}^{\alpha}_{0+}v_0(t) + (A + CI)v_0(t) = \lambda_1\sigma e_1 + C\sigma e_1 \leqslant f(t, \sigma e_1) + C\sigma e_1. \tag{11}$$

Since $\widetilde{S}_\alpha(t)$, $\widetilde{K}_\alpha(t)$ and $\widetilde{T}_\alpha(t)$ are positive α-resolvent operators and Λ is an increasing operator, so from Eq.(10)*, yields*

$$\begin{aligned}\sigma e_1 = v_0(t) &= \widetilde{S}_\alpha(t-t_0)v_0(t_0) + \widetilde{K}_\alpha(t-t_0)v_0(t_0) + \int_{t_0}^{t} \widetilde{T}_\alpha(t-s)\varphi(s)ds \\ &\leqslant \widetilde{S}_\alpha(t-t_0)x_0 + \widetilde{K}_\alpha(t-t_0)v_0(t_0) + \int_{t_0}^{t} \widetilde{T}_\alpha(t-s)\Big[f(s,\sigma e_1) + C\sigma e_1\Big]ds = \\ &\leqslant (\Lambda(\sigma e_1))(t).\end{aligned}$$

Note that $\sigma e_1 \leqslant u(t)$ $(\forall t \in I)$, then

$$\sigma e_1 \leqslant \Big(\Lambda(\sigma e_1)\Big)(t) \leqslant (\Lambda u)(t), \quad t \in I.$$

With this, $\Lambda : \Omega \to \Omega$ is continuous and increasing. We will show that the set $\Lambda(\Omega)$ is a family of equicontinuous functions in $C(I, E^+)$, using the monotonous iterative method.

Let $v_0 = \sigma e_1 \in \Omega$ and define the sequence $\{v_n\}$ by iteration

$$v_n = \Lambda v_{n-1}, \quad n = 1, 2, ... \tag{12}$$

Since Λ is an increasing operator and $v_1 = \Lambda v_0 \geqslant v_0$, we have the monotonous sequence,

$$v_0 \leqslant v_1 \leqslant v_2 \leqslant \cdots \leqslant v_n \leqslant \cdots \tag{13}$$

Therefore, $\{v_n\} = \{\Lambda v_{n-1}\} \subset \Lambda(\Omega) \subset \Omega$ is bounded and equicontinuous.

Let $B = \{v_n;\ n \in \mathbb{N}\}$ and $B_0 = \{v_{n-1};\ n \in \mathbb{N}\}$, then $B_0 = B \cup \{v_0\}$ and by noncompactness measurement property, we have $\mu(B(t)) = \mu(\Lambda(B_0)(t))$ for $t \in I$.

Substituting $\Lambda(B_0)(t)$, defined by Eq.(10)*, yields*

$$\mu(B(t)) = \mu\left(\left\{\int_{t_0}^{t} \widetilde{T}_\alpha(t-s)\Big[f(s, v_{n-1}(s)) + Cv_{n-1}(s)\Big]ds;\ n \in \mathbb{N}\right\}\right),$$

since, by item (3) *of the* Lemma 2.7*, we have to $\mu\Big(\widetilde{S}_\alpha(t-t_0)x_0 + \widetilde{K}_\alpha(t-t_0)x_1\Big) = 0$.*

Using the Lemma 2.11*, yields*

$$\begin{aligned}\mu(B(t)) &\leqslant 2\int_{t_0}^{t} \mu\left(\left\{\widetilde{T}_\alpha(t-s)\Big[f(s, v_{n-1}(s)) + Cv_{n-1}(s)\Big];\ n \in \mathbb{N}\right\}\right) ds \\ &\leqslant 2\int_{t_0}^{t} \|\widetilde{T}_\alpha(t-s)\| \cdot \mu\left(\left\{f(s, v_{n-1}(s)) + Cv_{n-1}(s);\ n \in \mathbb{N}\right\}\right) ds.\end{aligned}$$

As $(t-s)^{2-\alpha}\ \|\widetilde{T}_\alpha(t-s)\| \leqslant \overline{M}_{t_0}$ and using the condition (H_3)*, for any $t \in I$, yields*

$$\begin{aligned}\mu\Big(B(t)\Big) &\leqslant 2\overline{M}_{t_0}\int_{t_0}^{t} (t-s)^{\alpha-2}\left[L\mu\Big(B_0(s)\Big) + C\mu\Big(B_0(s)\Big)\right] ds \\ &\leqslant 2\overline{M}_{t_0}(L+C)\int_{t_0}^{t} (t-s)^{\alpha-2}\mu(B_0(s))ds \\ &\leqslant 0 + 2\overline{M}_{t_0}(L+C)\int_{t_0}^{t} (t-s)^{\alpha-2}\mu(B_0(s))ds.\end{aligned}$$

By the generalized inequality of Gronwall for fractional integral, Lemma 2.3*, yields*

$$\mu\Big(B(t)\Big) \leqslant 0\mathbb{E}_\alpha\Big(2\overline{M}(L+C)\Gamma(\alpha)(t-s)^\alpha\Big) = 0.$$

Hence, $\mu(B(t)) \equiv 0$ *for* $t \in I$*. The* Lemma 2.9 *says that* $\mu(B) = \max_{t\in I}\mu(B(t)) = 0$*, that is,* $\{v_n\}$ *is relatively compact in* $C(I, E^+)$*. Therefore, there is a subsequence* $\{v_{n_k}\} \subset \{v_n\}$ *such that* $v_{n_k} \xrightarrow{k\to\infty} u^* \in \Omega$*. Combining this with the sequence in* (13) *and the normality of the cone* E^+*, it is easy to see that* $v_n \xrightarrow{n\to\infty} u^*$*. Make* $n \to \infty$ *in* Eq.(12)*. From the continuity of the operator* Λ*, we have that* $u^* = \Lambda u^*$ *is a fixed point.*

Therefore, $u^* \in \Omega \subset C(I, E^+)$ *is a e-positive mild solution of Eq.*(8)*.*

(B) *The global existence of mild solutions for the system* (7) *in* $J_0 = [0, t_1]$*.*

*We saw in item **(A)** that the system* (7) *has an e-positive mild solution* $u_0 \in C([0, h_0], E^+)$ *expressed by*

$$u_0(t) = \widetilde{S}_\alpha(t)x_0 + \widetilde{K}_\alpha(t)x_1 + \int_0^t \widetilde{T}_\alpha(t-s)\Big[f(s,u_0(s)) + Cu_0(s)\Big]ds.$$

By the extension theorem, (see (PAZY, 2012))*,* u_0 *can be extended to a mild solution of the system* (7)*, which is also denoted by* $u_0 \in C([0,T), E^+)$*, whose existence interval is* $[0, T)$*.*

Now, we prove that $T > t_1$*. So consider* $\overline{a} = \max_{t\in[0,T+1]} a(t)$*,* $\overline{b} = \max_{t\in[0,T+1]} b(t)$*,* $M_1 = \sup\Big\{\|(t-T)^{2-\alpha}S_\alpha(t)\|;\ t \in [0,\ T+1]\Big\}$*,* $\widetilde{M_1} = \sup\Big\{\|(t-T)^{2-\alpha}K_\alpha(t)\|;\ t \in [0,\ T+1]\Big\}$*, and* $\overline{M}_1 = \sup\Big\{\|(t-T)^{2-\alpha}T_\alpha(t)\|;\ t \in [0,\ T+1]\Big\}$*.*

Suppose $T \leqslant t_1$ *and plotting the norm of the solution* u_0 *and using the condition* (H_1)*, yields*

$$\begin{aligned}
\|u_0(t)\| &\leqslant M_1\|x_0\| + \widetilde{M_1}\|x_1\| + \overline{M}_1\int_0^t (t-s)^{\alpha-2}\Big\|\Big[f(s,u_0(s)) + Cu_0(s)\Big]\Big\|ds\\
&\leqslant M_1\|x_0\| + \widetilde{M_1}\|x_1\| + \overline{M}_1\overline{b}\int_0^t (t-s)^{\alpha-2}ds + \overline{M}_1(\overline{a}+C)\int_0^t (t-s)^{\alpha-2}\|u_0(s)\|ds\\
&\leqslant M_1\|x_0\| + \widetilde{M_1}\|x_1\| + \overline{M}_1\overline{b}\,\frac{T^\alpha}{\alpha} + \overline{M}_1(\overline{a}+C)\int_0^t (t-s)^{\alpha-2}\|u_0(s)\|ds.
\end{aligned}$$

By the Gronwall generalized inequality for fractional integral, Corollary 2.3, we have

$$\begin{aligned}
\|u_0(t)\| &\leqslant \left(M_1\|x_0\| + \widetilde{M_1}\|x_1\| + \overline{M}_1\overline{b}\,\frac{T^\alpha}{\alpha}\right)\mathbb{E}_\alpha\Big(M_1(\overline{a}+C)\Gamma(\alpha)t\Big)\\
&\leqslant \left(M_1\|x_0\| + \overline{M}_1\overline{b}\,\frac{T^\alpha}{\alpha}\right)\mathbb{E}_\alpha\Big(M_1(\overline{a}+C)\Gamma(\alpha)T\Big) := M_2.
\end{aligned} \tag{14}$$

Define the constant

$$N_0 := \sup\Big\{\|f(t,x)\|;\ t \in [0,\ T+1]\ e\ \|x\| \leqslant M_2\Big\}. \tag{15}$$

As $\widetilde{S}_\alpha(t)$ *and* $\widetilde{K}_\alpha(t)$ *are continuous in the operator's norm for* $t > 0$*, for any* $0 < \tau_1 < \tau_2 < T$*, we consider the following functions*

$$u_0(\tau_2) = \widetilde{S}_\alpha(\tau_2)x_0 + \widetilde{K}_\alpha(\tau_2)x_1 + \int_0^{\tau_2}\widetilde{T}_\alpha(\tau_2-s)\Big(f(s,u_0(s)) + Cu_0(s)\Big)ds. \tag{16}$$

$$u_0(\tau_1) = \widetilde{S}_\alpha(\tau_1)x_0 + \widetilde{K}_\alpha(\tau_1)x_1 + \int_0^{\tau_1}\widetilde{T}_\alpha(\tau_1-s)\Big(f(s,u_0(s)) + Cu_0(s)\Big)ds. \tag{17}$$

Subtracting Eq.(17) *from* Eq.(16), *yields*

$$u_0(\tau_2) - u_0(\tau_1) = \widetilde{S}_\alpha(\tau_2)x_0 - \widetilde{S}_\alpha(\tau_1)x_0 + \widetilde{K}_\alpha(\tau_2)x_1 - \widetilde{K}_\alpha(\tau_1)x_1 + \int_0^{\tau_2} \widetilde{T}_\alpha(\tau_2 - s)\Big[f(s, u_0(s)) + Cu_0(s)\Big]ds$$
$$- \int_0^{\tau_1} \widetilde{T}_\alpha(\tau_1 - s)\Big[f(s, u_0(s)) + Cu_0(s)\Big]ds.$$

Rearranging integrals with respect to integration limits, yields

$$\begin{aligned}\|u_0(\tau_2) - u_0(\tau_1)\| \quad &\leqslant \|\widetilde{S}_\alpha(\tau_2)x_0 - \widetilde{S}_\alpha(\tau_1)x_0\| + \|\widetilde{K}_\alpha(\tau_2)x_1 - \widetilde{K}_\alpha(\tau_1)x_1\| \\ &+ \int_0^{\tau_1} \|\widetilde{T}_\alpha(\tau_2 - s) - \widetilde{T}_\alpha(\tau_1 - s)\|\|f(s, u_0(s)) + Cu_0(s)\|ds \\ &+ \int_{\tau_1}^{\tau_2} \|\widetilde{T}_\alpha(\tau_2 - s)\|\|f(s, u_0(s)) + Cu_0(s)\|ds.\end{aligned}$$

Changing the variable, s to $\tau_1 - s$, in the first integral and using the Eq.(15)*, the* Eq.(14) *and the constant $\overline{M}_1$, yields*

$$\begin{aligned}\|u_0(\tau_2) - u_0(\tau_1)\| \quad &\leqslant \|\widetilde{S}_\alpha(\tau_2)x_0 - \widetilde{S}_\alpha(\tau_1)x_0\| + \|\widetilde{K}_\alpha(\tau_2)x_1 - \widetilde{K}_\alpha(\tau_1)x_1\| \\ &+(N_0 + CM_2)\int_0^{\tau_1} \|\widetilde{T}_\alpha(\tau_2 - \tau_1 + s) - \widetilde{T}_\alpha(s)\|ds \\ &+\overline{M}_1(N_0 + CM_2)\int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-2}ds.\end{aligned}$$

From this it follows that

$$\begin{aligned}\|u_0(\tau_2) - u_0(\tau_1)\| \quad &\leqslant \|\widetilde{S}_\alpha(\tau_2)x_0 - \widetilde{S}_\alpha(\tau_1)x_0\| + \|\widetilde{K}_\alpha(\tau_2)x_1 - \widetilde{K}_\alpha(\tau_1)x_1\| + \overline{M}_1(N_0 + CM_2)\frac{(\tau_2 - \tau_1)^\alpha}{\alpha} \\ &+(N_0 + CM_2)\int_0^{\tau_1} \|\widetilde{T}_\alpha(\tau_2 - \tau_1 + s) - \widetilde{T}_\alpha(s)\|ds.\end{aligned}$$

When $\tau_1 \to T^-$ e $\tau_2 \to T^-$, yields

$$\begin{cases}\|\widetilde{S}^*_\alpha(\tau_2)x_0 - \widetilde{S}^*_\alpha(\tau_1)x_0\| \xrightarrow{\tau_1,\tau_2 \to T^-} 0, \\ \|\widetilde{K}^*_\alpha(\tau_2)x_1 - \widetilde{K}^*_\alpha(\tau_1)x_1\| \xrightarrow{\tau_1,\tau_2 \to T^-} 0, \\ \dfrac{(\tau_2 - \tau_1)^\alpha}{\alpha} \xrightarrow{\tau_1,\tau_2 \to T^-} 0, \\ \int_0^T \|\widetilde{T}^*_\alpha(\tau_2 - \tau_1 + s) - \widetilde{T}^*_\alpha(s)\|ds \xrightarrow{\tau_1,\tau_2 \to T^-} 0.\end{cases}$$

Thus $\|u_0(\tau_2) - u_0(\tau_1)\| \equiv 0$. Using the Cauchy criterion, there is $\overline{x} \in E^+$ such that $\lim\limits_{t \to T^-} u_0(t) = \overline{x}$.

We consider the system with evolution fractional equation and without impulse in E, given by

$$\begin{cases}{}^c\mathbb{D}^\alpha_{0+}u(t) + (A + CI)u(t) = f(t, u(t)) + Cu(t), \quad t > T, \\ u(T) = \overline{x} \text{ and } u'(T) = \overline{y}.\end{cases} \tag{18}$$

Using the **(A)** *part, the system* (18) *has an e-positive mild solution v in $[T,\ T + h_T]$. Let*

$$\overline{u}(t) = \begin{cases}u_0(t), & t \in [0, T), \\ v(t), & t \in [T,\ T + h_T].\end{cases}$$

It is easy to see that $\overline{u}(t)$ is an e-positive mild solution of the system (7) *in* $[0, T + h_T]$. *Therefore, $\overline{u}(t)$ is an extension of $u_0(t)$, this is a contradiction. Thus, $T > t_1$, that is, the global e-positive mild solution $u_0(t)$ of the system* (7) *exists in J_0, which is also an e-positive mild solution of the system* (P) *in J_0.*

(II) *In this second part, we will prove the global existence of e-positive mild solutions in the interval J_∞.*

Initially, we will prove that the system (P) *has a global e-positive mild solution in $J_1 = (t_1, t_2]$. We consider the system with evolution fractional equation without impulse in J_1*

$$\begin{cases} {}^{C}\mathbb{D}^{\alpha,\beta}_{0+}u(t) + (A + CI)u(t) = f(t, u(t)) + Cu(t), \quad t \in J_1, \\ u(t_1^+) = u_0(t_1) + I(u_0(t_1)), \\ u'(t_1^+) = u_0'(t_1) + I(u_0'(t_1)). \end{cases} \tag{19}$$

Clearly, a global e-positive mild solution of the system (19) *in J_1 is also an e-positive mild solution of the system* (P) *in J_1. From the proof of item (I), for $t \in J_0 = [0, t_1]$, we have*

$$u_0(t) = \widetilde{S}_\alpha(t)x_0 + \widetilde{K}_\alpha(t)x_1 + \int_0^t \widetilde{T}_\alpha(t-s)\Big[f(s, u(s)) + Cu(s)\Big]ds. \tag{20}$$

By an argument similar to the proof of (I), *the system* (19) *has an e-positive mild solution $u_1 \in C(J_1, E^+)$, with $J_1 = (t_1, t_2]$, given by*

$$u_1(t) = \widetilde{S}_\alpha(t)\theta_0 + \widetilde{K}_\alpha(t)\bar{\theta}_0 + \int_0^t \widetilde{T}_\alpha(t-s)\Big[f(s, u(s)) + Cu(s)\Big]ds. \tag{21}$$

By the impulsive condition and Eq.(20) *and* Eq.(21), *yields*

$$\begin{cases} \theta_0 = x_0 + \widetilde{S}_\alpha^{-1}(t_1)\, I\Big(u_0(t_1)\Big). \\ \bar{\theta}_0 = x_1 + \widetilde{K}_\alpha^{-1}(t_1)\, I\Big(u_0'(t_1)\Big). \end{cases} \tag{22}$$

Then, for $t \in J_1 = (t_1, t_2]$, we have

$$\begin{aligned} u_1(t) = \widetilde{S}_\alpha(t)x_0 + \widetilde{K}_\alpha(t)x_1 + \int_0^t \widetilde{T}_\alpha(t-s)\Big[f(s, u(s)) + Cu(s)\Big]ds \; + \\ + \widetilde{S}_\alpha(t)\widetilde{S}_\alpha^{-1}(t_1)\, I\Big(u_0(t_1)\Big) + \widetilde{K}_\alpha(t)\widetilde{K}_\alpha^{-1}(t_1)\, I\Big(u_0'(t_1)\Big). \end{aligned} \tag{23}$$

Now consider $J_2 = (t_2, t_3]$ and $u_2 \in C(J_2, E^+)$, follow

$$u_2(t) = \widetilde{S}_\alpha(t)\theta_1 + \widetilde{K}_\alpha(t)\bar{\theta}_1 + \int_0^t \widetilde{T}_\alpha(t-s)\Big[f(s, u(s)) + Cu(s)\Big]ds. \tag{24}$$

By the impulsive condition and Eq.(23) *and* Eq.(24), *yields*

$$\begin{cases} \theta_1 = x_0 + \widetilde{S}_\alpha^{-1}(t_1)\, I\Big(u_0(t_1)\Big) + \widetilde{S}_\alpha^{-1}(t_2)\, I\Big(u_1(t_2)\Big). \\ \bar{\theta}_1 = x_1 + \widetilde{K}_\alpha^{-1}(t_1)\, I\Big(u_0'(t_1)\Big) + \widetilde{K}_\alpha^{-1}(t_2)\, I\Big(u_1'(t_2)\Big). \end{cases} \tag{25}$$

So, for $t \in J_2 = (t_2, t_3]$, we have

$$u_2(t) = \widetilde{S}_\alpha(t)\theta_1 + \widetilde{K}_\alpha(t)\bar{\theta}_1 + \int_0^t \widetilde{T}_\alpha(t-s)\Big[f(s,u(s)) + Cu(s)\Big]ds \; + \\ + \widetilde{S}_\alpha(t)\widetilde{S}_\alpha^{-1}(t_1)\, I\Big(u_0(t_1)\Big) + \widetilde{S}_\alpha(t)\widetilde{S}_\alpha^{-1}(t_2)\, I\Big(u_1(t_2)\Big) + \\ + \widetilde{K}_\alpha(t)\widetilde{K}_\alpha^{-1}(t_1)\, I\Big(u_0'(t_1)\Big) + \widetilde{K}_\alpha(t)\widetilde{K}_\alpha^{-1}(t_2)\, I\Big(u_1'(t_2)\Big).$$

Suppose that, for $t \in J_{k-1}$ $(k = 4, 5, \ldots)$, the system (P) has an e-positive mild solution $u_{k-1} \in C(J_{k-1}, E^+)$ $(k = 4, 5, \ldots)$. So, for $t \in J_k$ $(k = 3, 4, \ldots)$, the system with evolution fractional equation without impulse in E

$$\begin{cases} {}^C\mathbb{D}_{0+}^\alpha u(t) + (A + CI)u(t) = f(t, u(t)) + Cu(t), & t \in J_k,\ k = 3, 4, \ldots \\ u(t_k^+) = u_{k-1}(t_k) + I(u_{k-1}(t_k)), \\ u'(t_k^+) = u'_{k-1}(t_k) + I(u'_{k-1}(t_k)), \end{cases} \tag{26}$$

has e-positive mild solution $u_k \in C(J_k, E^+)$, expressed by

$$\begin{aligned} u_k(t) &= \widetilde{S}_\alpha(t)\theta_{k-1} + \widetilde{K}_\alpha(t)\bar{\theta}_{k-1} + \int_{t_k}^t \widetilde{T}_\alpha(t-s)\Big[f(s,u(s)) + Cu(s)\Big]ds \\ &= \widetilde{S}_\alpha(t)\Big[x_0 + \widetilde{S}_\alpha^{-1}(t_1)\, I\Big(u_0(t_1)\Big) + \widetilde{S}_\alpha^{-1}(t_2)\, I\Big(u_1(t_2)\Big) + \cdots + \widetilde{S}_\alpha^{-1}(t_k) I\Big(u_{k-1}(t_k)\Big)\Big] \\ &+ \widetilde{K}_\alpha(t)\Big[x_1 + \widetilde{K}_\alpha^{-1}(t_1)\, I\Big(u_0'(t_1)\Big) + \widetilde{K}_\alpha^{-1}(t_2)\, I\Big(u_1'(t_2)\Big) + \cdots + \widetilde{K}_\alpha^{-1}(t_k) I\Big(u_{k-1}'(t_k)\Big)\Big] \\ &+ \int_{t_k}^t \widetilde{T}_\alpha(t-s)\Big[f(s,u(s)) + Cu(s)\Big]ds \end{aligned}$$

...

$$u_k(t) = \widetilde{S}_\alpha(t)x_0 + \widetilde{K}_\alpha(t)x_1 + \int_0^t \widetilde{T}_\alpha(t-s)\Big[f(s,u(s)) + Cu(s)\Big]ds \\ + \widetilde{S}(t)\sum_{j=1}^k \widetilde{S}_\alpha^{-1}(t_j) I(u_{j-1}(t_j)) + \widetilde{K}(t)\sum_{j=1}^k \widetilde{K}_\alpha^{-1}(t_j) I(u'_{j-1}(t_j)). \tag{27}$$

Now, we define a u function as

$$u(t) = \begin{cases} u_0(t), & t \in J_0, \\ u_1(t), & t \in J_1, \\ \cdots \\ u_k(t), & t \in J_k\ (k = 2, 3, \ldots), \end{cases} \tag{28}$$

It is clear that $u(t) \in PC(J_\infty, E^+)$ is an e-positive mild solution of the system (P), which satisfies

$$u_k(t) = \widetilde{S}_\alpha(t)x_0 + \widetilde{K}_\alpha(t)x_1 + \int_0^t \widetilde{T}_\alpha(t-s)\Big[f(s,u(s)) + Cu(s)\Big]ds \\ + \widetilde{S}_\alpha(t)\sum_{i=1}^k \widetilde{S}_\alpha^{-1}(t_i) I(u_(t_i)) + \widetilde{K}_\alpha(t)\sum_{i=1}^k \widetilde{K}_\alpha^{-1}(t_i) I(u'(t_i)). \tag{29}$$

By the global existence property of $u_i(t)$ in J_i, $i \in \mathbb{N}$, the solution $u(t)$ defined by Eq.(28) is a global e-positive mild solution of the system (P) in J_∞.

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