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Continuation of mild solution for abstract fractional integro-differential equations

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Abstract

In this paper we study the existence of local, global mild solution for the abstract fractional integro-differential Cauchy problem

$$D_t^\alpha u(t) = Au(t) + \int_0^t B(t-s)u(s)ds + f(t,u(t)), \quad (1)$$

$$u(0) = u_0 \in X, \quad (2)$$

where $D_t^\alpha u$ represents the Caputo derivative for $\alpha \in (0,1)$, A, $(B(t))_{t\geq 0}$ are closed linear operators defined on a common domain which is dense in a Banach space X and f satisfies appropriated conditions.

Keywords: Integro-differential equations. resolvent of operators. fractional differential equation. global mild solution.

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1 Introdução

In this paper we study the existence of local and global solution for the abstract fractional integro-differential system. For this purpose, we introduce the theory of resolvent operator studied in (SANTOS, 2019) for the fractional integro-differential problem

$$D_t^{\alpha} u(t) = Au(t) + \int_0^t B(t-s)u(s)ds, \quad t \geq 0, \tag{3}$$

$$u(0) = u_0. \tag{4}$$

where $A, (B(t))_{t\geq 0}$ are closed linear operators defined on a common domain which is dense in a Banach space $(X, \|\cdot\|)$, and $D_t^{\alpha} h(t)$ represents the Caputo derivative of $\alpha \in (0,1)$ defined by

$$D_t^{\alpha} h(t) := \int_0^t g_{1-\alpha}(t-s)h'(s)ds,$$

where $g_{1-\alpha}$ is the Gelfand-Shilov function $g_{\beta}(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}, t > 0$, with $\beta = 1-\alpha$.

In the past decades, considerable attention has been attracted to the theory of resolvent operator for integro-differential equations. We refer to the book by Gripenberg et. al. (GRIPENBERG; LONDEN; STAFFANS, 1990) for the case where the underlying space X has finite dimension. For abstract integro-differential equations on infinite dimensional spaces, we cite the book by Prüss (PRÜSS, 2013) and the papers of Da Prato et al. (DA PRATO; IANNELLI, 1985; DA PRATO; LUNARDI, 1988), Grimmer et al. (GRIMMER; KAPPEL, 1984; GRIMMER; PRITCHARD, 1983; GRIMMER; PRÜSS, 1985), Lunardi (LUNARDI, 1990, 1985), Sforza (SFORZA, 1991) and Dos Santos et al. (SANTOS; HENRÍQUEZ, 2015; SANTOS; HENRÍQUEZ; HERNÁNDEZ, 2011). With a resolvent family also it is possible study a existence and regularity of solutions for fractional integro-differential equations (AGARWAL; SANTOS; CUEVAS, 2012; LI; SUN; FENG, 2016a).

Regarding the fractional differential equations in spaces of infinite dimension this problem has been extensive studied, we can mention the pioner thesis of Bajlekova (BAJLEKOVA et al., 2001) and the works of (HERNÁNDEZ; O'REGAN; BALACHANDRAN, 2013; LI; SUN; FENG, 2016b; KEYANTUO; LIZAMA; WARMA, 2013; LI; SUN; FENG, 2016a; WANG; CHEN; XIAO, 2012; ZHOU; JIAO, 2010) and references therein. For abstract fractional integro-differential equations in infinite dimension we suggest the articles Agarwal et. al. (AGARWAL; SANTOS; CUEVAS, 2012) in the case of $\alpha \in (1,2)$, the book of Kostić (KOSTIC, 2015), Ponce (PONCE, 2013) and Herzallah et. al. (EL-SAYED; HERZALLAH, 2005) when $B(t) = a(t)A, t \geq 0$. To the best of the authors' knowledge, a continuation solutions theorem of mild solutions for the (3)-(4) with $\alpha \in (0,1)$ is a subject that has not been treated in the literature. This is the principal motivation of this paper.

This work has three Sections. In Section 2, we comment about the theory of α-resolvent operator introduced in (SANTOS, 2019) for the better understanding of work. In Section 3, we show the existence of local, global existence and uniqueness of mild solution for the non-homogeneous equation (1)-(2) is discussed.

By $D_t^{\alpha} h(t)$ we denoted the Caputo derivative of $\alpha > 0$ defined by (SAMKO; KILBAS; MARICHEV, 1993)

$$D_t^{\alpha} h(t) := \int_0^t g_{n-\alpha}(t-s)\frac{d^n}{ds^n}h(s)ds,$$

where n is the smallest integer greater than or equal to α and $g_{\beta}(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}, t > 0, \beta \geq 0$. These functions satisfy the semigroup property

$$g_{\alpha} * g_{\beta} = g_{\alpha+\beta}.$$

If we denote

$$J_t^\alpha f(t) = (g_\alpha * f)(t) = \int_0^t g_\alpha(t-s)f(s)ds, \tag{5}$$

we have

$$\begin{aligned} D_t^\alpha J_t^\alpha f(t) &= f(t), & (6)\\ J_t^\alpha D_t^\alpha f(t) &= f(t) - \sum_{k=0}^{n-1} f^{(k)}(0)\frac{t^k}{k!}. & (7)\end{aligned}$$

Applying the properties of the Laplace transform and taking into account that $\widehat{g_\alpha}(\lambda) = \lambda^{-\alpha}$, we obtain

$$\widehat{D_t^\alpha} f(\lambda) = \lambda^\alpha f(\lambda) - \sum_{k=0}^{n-1} f^{(k)}(0)\lambda^{\alpha-1-k}, \tag{8}$$

(see (BAJLEKOVA et al., 2001; SAMKO; KILBAS; MARICHEV, 1993) for details.)

Throughout this paper, let $(Z, \|\cdot\|_Z)$ and $(W, \|\cdot\|_W)$ be Banach spaces. We denote by $\mathcal{L}(Z, W)$ the space of bounded linear operators from Z into W endowed with norm of operators, and we write simply $\mathcal{L}(Z)$ when $Z = W$. By $\mathbf{R}(Q)$ we denote the range of a map Q and for a closed linear operator $P : D(P) \subseteq Z \to W$, the notation $[D(P)]$ represents the domain of P endowed with the graph norm, $\|z\|_1 = \|z\|_Z + \|Pz\|_W$, $z \in D(P)$. The notation, $B(x, R)$ and $B[x, R]$ represent the open ball and the closed ball respectively with center at x and radius $R > 0$ in X. Let $I \subset \mathbb{R}$, by $C(I, X)$ we denote the space of continuous functions defined on I into X and $C^1(I, X)$ stands for the space of continuous functions from I to X having continuous derivative. We define the space $C^\alpha(I, X)$, by

$$C^\alpha(I, X) := \{x \in C(I, X) : D_t^\alpha x \in C(I, X)\}.$$

We denote by $L^p(I, X)$ the set of all meansurable functions $u(\cdot)$ on I, into X such that $\| u(t) \|^p$ is integrable ans its norm is given by $\| u \|_{L^p(I,X)} = \left(\int_I \| u(t) \|^p\right)^{\frac{1}{p}}$; similarly, by $L^p_{loc}(\mathbb{R}_+, X)$ we denote the space of the functions belonging $L^p(I, X)$ for any compact set $I \subset \mathbb{R}_+$. When $X = \mathbb{R}^n$, for some n, we denote for simplicity by $C(I), C^1(I), C^\alpha(I), L^p(I)$ and $L^p_{loc}(\mathbb{R}_+)$ respectively. The notation $\rho(P)$ stands for the resolvent set of P and $R(\lambda, P) = (\lambda I - P)^{-1}$ is the resolvent operator of P. Furthermore, for appropriate functions $K : [0, \infty) \to Z$ and $S : [0, \infty) \to \mathcal{L}(Z, W)$, the notation $\widehat{K}$ denotes the Laplace transform of K and $S * K$ the convolution between S and K, which is defined by $S * K(t) = \int_0^t S(t-s)K(s)ds$.

2 Preliminaries

To begin, we introduce the following concept of resolvent operator for the abstract fractional integro-differential problem (3)-(4).

Definition 1 *A one parameter family of bounded linear operators $(\mathcal{R}_\alpha(t))_{t\geq 0}$ on X is called a α-resolvent operator of (3)-(4) if the following conditions are verified.*

(a) The function $\mathcal{R}_\alpha(\cdot) : [0, \infty) \to \mathcal{L}(X)$ is strongly continuous and $\mathcal{R}_\alpha(0)x = x$ for all $x \in X$ and $\alpha \in (0, 1)$.

(b) *For $x \in D(A)$, $\mathcal{R}_\alpha(\cdot)x \in C([0,\infty),[D(A)]) \bigcap C^\alpha((0,\infty),X)$, and*

$$D_t^\alpha \mathcal{R}_\alpha(t)x = A\mathcal{R}_\alpha(t)x + \int_0^t B(t-s)\mathcal{R}_\alpha(s)xds \tag{9}$$

$$= \mathcal{R}_\alpha(t)Ax + \int_0^t \mathcal{R}_\alpha(t-s)B(s)xds, \tag{10}$$

for every $t \geq 0$.

In this work we always assume that the following conditions are verified.

(H1) The operator $A : D(A) \subseteq X \to X$ is a closed linear operator with $[D(A)]$ dense in X, for some $\phi \in (\frac{\pi}{2}, \pi)$ there is positive constants $C_0 = C_0(\phi)$ such that $\lambda \in \rho(A)$ for each

$$\Sigma_{0,\phi} = \{\lambda \in \mathbb{C} :\mid arg(\lambda) \mid < \phi\} \subset \rho(A),$$

and $\| R(\lambda, A) \| \leq \dfrac{C_0}{\mid \lambda \mid}$ for all $\lambda \in \Sigma_{0,\phi}$.

(H2) For all $t \geq 0$, $B(t) : D(B(t)) \subseteq X \to X$ is a closed linear operator, $D(A) \subseteq D(B(t))$ and $B(\cdot)x$ is strongly measurable on $(0,\infty)$ for each $x \in D(A)$. There exists $b(\cdot) \in L^1_{loc}(\mathbb{R}^+)$ such that $\widehat{b}(\lambda)$ exists for $Re(\lambda) > 0$ and $\| B(t)x \| \leq b(t) \| x \|_1$ for all $t > 0$ and $x \in D(A)$. Moreover, the operator valued function $\widehat{B} : \Sigma_{0,\pi/2} \to \mathcal{L}([D(A)], X)$ has an analytical extension (still denoted by $\widehat{B}$) to $\Sigma_{0,\phi}$ such that $\|\widehat{B}(\lambda)x\| \leq \|\widehat{B}(\lambda)\| \, \|x\|_1$ for all $x \in D(A)$, and $\|\widehat{B}(\lambda)\| = O(\frac{1}{|\lambda|})$, as $|\lambda| \to \infty$.

(H3) There exists a subspace $D \subseteq D(A)$ dense in $[D(A)]$ and positive constants C_i, $i = 1, 2$, such that $A(D) \subseteq D(A)$, $\widehat{B}(\lambda)(D) \subseteq D(A)$, $\|A\widehat{B}(\lambda)x\| \leq C_1\|x\|$ for every $x \in D$ and all $\lambda \in \Sigma_{0,\phi}$.

Remark 1 *We note that conditions of type* (**H2**) *and* (**H3**) *have been previously considered in the literature; see (SANTOS; HENRÍQUEZ, 2015; SANTOS; HENRÍQUEZ; HERNÁNDEZ, 2011; GRIMMER; PRITCHARD, 1983) for details.*

In the sequel, for $r > 0$ and $\theta \in (\frac{\pi}{2}, \phi)$,

$$\Sigma_{r,\theta} = \{\lambda \in \mathbb{C} :\mid \lambda \mid \geq r, \text{ and } \mid arg(\lambda) \mid < \theta\}.$$

In addition, $\rho(F_\alpha)$ and $\rho(G_\alpha)$ are the sets

$$\rho(F_\alpha) = \{\lambda \in \mathbb{C} : F_\alpha(\lambda) := (\lambda^\alpha I - A - \widehat{B}(\lambda))^{-1} \in \mathcal{L}(X)\} \text{ and}$$

$$\rho(G_\alpha) = \{\lambda \in \mathbb{C} : G_\alpha(\lambda) := \lambda^{\alpha-1}(\lambda^\alpha I - A - \widehat{B}(\lambda))^{-1} \in \mathcal{L}(X)\}.$$

We next study some preliminary properties needed to establish existence of a α-resolvent operator for the problem (1)-(2). The proof of the nexts rsults can be found (SANTOS, 2019).

In the rest of this paper we assume the conditions (**Hi**), $i = 1, 2, 3$, holds, r, θ are numbers such that $r > r_1$ and $\theta \in (\pi/2, \phi)$. By $\Gamma_{r,\theta}, \Gamma^i_{r,\theta}$, $i = 1, 2, 3$, we define the paths

$$\Gamma^1_{r,\theta} = \{te^{i\theta} : t \geq r\}, \quad \Gamma^2_{r,\theta} = \{re^{i\xi} : -\theta \leq \xi \leq \theta\} \text{ and } \Gamma^3_{r,\theta} = \{te^{-i\theta} : t \geq r\},$$

and $\Gamma_{r,\theta} = \bigcup_{i=1}^3 \Gamma^i_{r,\theta}$ oriented counterclockwise.

Remark 2 *The properties about the families G_α and F_α were established in in (SANTOS, 2019).*

We start defining the α-resolvent families for the problem (3)-(4) with $\alpha \in (0,1)$.

Definition 2 *We define the operator family $(\mathcal{R}_\alpha(t))_{t\geq 0}$ by*

$$\mathcal{R}_\alpha(t) = \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t} G_\alpha(\lambda)d\lambda, t \geq 0, \tag{11}$$

and the auxiliary resolvent operator family $(\mathcal{S}_\alpha(t))_{t\geq 0}$ by

$$\mathcal{S}_\alpha(t) = \frac{t^{1-\alpha}}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t} F_\alpha(\lambda)d\lambda, t \geq 0. \tag{12}$$

Remark 3 *When $B(t) = 0$, for all $t \geq 0$, the operators family $(\mathcal{R}_\alpha(t))_{t\geq 0}$ and $(\mathcal{S}_\alpha(t))_{t\geq 0}$ coincide with operators family $(E_\alpha(t^\alpha A))_{t\geq 0}$ and $(E_{\alpha,\alpha}(t^\alpha A))_{t\geq 0}$ respectively, for more details by $(E_\alpha(t^\alpha A))_{t\geq 0}$ and $(E_{\alpha,\alpha}(t^\alpha A))_{t\geq 0}$ see (ANDRADE et al., 2015; BAJLEKOVA et al., 2001; CARVALHO NETO, 2013) and the references therein.*

We next will establish some properties of $(\mathcal{R}_\alpha(t))_{t\geq 0}$ and $(\mathcal{S}_\alpha(t))_{t\geq 0}$ family.

Theorem 1 *The operator function $\mathcal{R}_\alpha(\cdot)$ is:*

(i) exponentially bounded in $\mathcal{L}(X)$;

(ii) exponentially bounded in $\mathcal{L}([D(A)])$;

(iii) strongly continuous on $[0,\infty)$ and uniformly continuous on $(0,\infty)$;

(iv) strongly continuous on $[0,\infty)$ in $\mathcal{L}([D(A)])$.

Comment: Proof of (i). If $t > 1$, from (11) we get

$$\begin{aligned}
\| \mathcal{R}_\alpha(t) \| &= \| \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t} G_\alpha(\lambda)d\lambda \| \\
&\leq \frac{C}{\pi}\int_r^\infty e^{ts\cos\theta}\frac{ds}{s} + \frac{C}{2\pi}\int_{-\theta}^{\theta} e^{tr\cos\xi}d\xi \\
&\leq \left(\frac{C}{\pi r|\cos\theta|} + \frac{C\theta}{\pi}\right)e^{rt}.
\end{aligned}$$

If $t \in (0,1)$, using that $G_\alpha(\cdot)$ is analytic on $\Sigma_{r,\theta}$, we get

$$\begin{aligned}
\| \mathcal{R}_\alpha(t) \| &= \| \frac{1}{2\pi i}\int_{\Gamma_{\frac{r}{t},\theta}} e^{\lambda t} G_\alpha(\lambda)d\lambda \| \\
&\leq \frac{C}{\pi}\int_{\frac{r}{t}}^\infty e^{ts\cos\theta}\frac{ds}{s} + \frac{C}{2\pi}\int_{-\theta}^{\theta} e^{r\cos\xi}d\xi \\
&\leq \left(\frac{C}{\pi}\int_r^\infty e^{u\cos\theta}\frac{du}{u} + \frac{C}{2\pi}\int_{-\theta}^{\theta} e^{r\cos\xi}d\xi\right) \\
&\leq \left(\frac{C}{\pi r|\cos\theta|} + \frac{C\theta}{\pi}\right)e^{r}.
\end{aligned}$$

This shows (i)

Proof of (ii). From (11) that the integral in

$$R(t)=\frac{1}{2\pi i}\int_{\Gamma_{r,\,\theta}} e^{\lambda t}AG_\alpha(\lambda)d\lambda,\ t>0,$$

is absolutely convergent in $\mathcal{L}([D(A)],X)$ and defines a linear operator

$$R(t)\in\mathcal{L}([D(A)],X).$$

Using that A is closed, we can affirm that $R(t)=A\mathcal{R}_\alpha(t)$.

From Lemma, $G_\alpha:\Sigma_{r,\vartheta}\to\mathcal{L}([D(A)])$ is analytic and $\| G_\alpha(\lambda)\|_1\leq C|\lambda|^{-1}$. If $t>1$ and $x\in D(A)$, we get

$$\begin{aligned}\| A\mathcal{R}_\alpha(t)x\| &= \|\frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t}AG_\alpha(\lambda)xd\lambda\|\\ &\leq \left(\frac{C}{\pi}\int_r^\infty e^{ts\cos\theta}\frac{ds}{s}+\frac{C}{2\pi}\int_{-\theta}^{\theta}e^{tr\cos\xi}d\xi\right)\| x\|_1\\ &\leq \left(\frac{C}{\pi r|\cos\theta|}+\frac{C\theta}{\pi}\right)e^{rt}\| x\|_1\ .\end{aligned}$$

For $t\in(0,1)$ and $x\in D(A)$ we get

$$\begin{aligned}\| A\mathcal{R}_\alpha(t)x\| &= \|\frac{1}{2\pi i}\int_{\Gamma_{\frac{r}{t},\theta}} e^{\lambda t}AG_\alpha(\lambda)xd\lambda\|\\ &\leq \frac{C}{\pi}\int_{\frac{r}{t}}^\infty e^{ts\cos\theta}\frac{ds}{s}\| x\|_1\\ &\quad+\frac{C}{2\pi}\int_{-\theta}^{\theta}e^{r\cos\xi}d\xi\| x\|_1\\ &\leq \left(\frac{C}{\pi r|\cos\theta|}+\frac{C\theta}{\pi}\right)e^{r}\| x\|_1\ .\end{aligned}$$

From before we obtain $\mathcal{R}(\cdot)$ is exponentially bounded in $\mathcal{L}([D(A)])$.

Proof of $(iii.)$ It is clear from (11) that $\mathcal{R}_\alpha(\cdot)x$ is uniformly and strongly continuous at $t>0$ for every $x\in X$. We next establish the strongly continuity at $t=0$. Using that

$$\frac{1}{2\pi i}\int_{\Gamma_{r,\,\theta}}\lambda^{-1}e^{\lambda t}d\lambda=\lim_{N\to\infty}\frac{1}{2\pi i}\int_{\{\Gamma_{r,\,\theta}:\ r\leq s\leq N\}\cup C_{N,\,\theta}}\lambda^{-1}e^{\lambda t}d\lambda=1,$$

where $C_{N,\theta}$ represent the curve $Ne^{i\xi}$ for $\theta\leq\xi\leq 2\pi-\theta$. For $x\in D(A)$ and $0<t\leq 1$ we get

$$\begin{aligned}\mathcal{R}_\alpha(t)x-x &= \frac{1}{2\pi i}\int_{\Gamma_{r,\,\theta}}\left(e^{\lambda t}G_\alpha(\lambda)x-\lambda^{-1}e^{\lambda t}x\right)d\lambda\\ &= \frac{1}{2\pi i}\int_{\Gamma_{r,\,\theta}} e^{\lambda t}\lambda^{-1}F_\alpha(\lambda)(A+\widehat{B}(\lambda))x\,d\lambda.\end{aligned}$$

Furthermore, it follows, and assumption **(H2)** that

$$\| e^{\lambda t}\lambda^{-1}F_\alpha(\lambda)(A+\widehat{B}(\lambda))x\|\leq e^rC\left(\frac{1}{|\lambda|^{\alpha+1}}\right)=H(\lambda),$$

where $H(\cdot)$ is integrable for $\lambda \in \Gamma_{r,\theta}$. From the Lebesgue dominated convergence theorem we infer that

$$\lim_{t\to 0^+} (\mathcal{R}_\alpha(t)x - x) = \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} \lambda^{-1} F_\alpha(\lambda)(A + \widehat{B}(\lambda))x d\lambda. \tag{13}$$

Let now $C_{L,\theta}$ be the curve $Le^{i\xi}$ for $-\theta \le \xi \le \theta$. Turning to apply the Cauchy's Theorem combining with the estimate

$$\| \int_{C_{L,\theta}} \lambda^{-1} F_\alpha(\lambda)(A + \widehat{B}(\lambda))x d\lambda \| \le \frac{C\theta}{L^\alpha}$$

we obtain

$$\begin{aligned} &\frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} \lambda^{-1} F_\alpha(\lambda)(A + \widehat{B}(\lambda))x d\lambda \\ &= \lim_{L\to\infty} \frac{1}{2\pi i}\int_{\{\Gamma_{r,\theta}:\ r\le s\le L\}\cup C_{L,\theta}} \lambda^{-1} F_\alpha(\lambda)(A + \widehat{B}(\lambda))x d\lambda = 0, \end{aligned}$$

we can affirm that $\lim_{t\to 0^+} \| \mathcal{R}_\alpha(t)x - x \| = 0$ for all $x \in D(A)$, which completes the proof of the strongly continuity on $\mathcal{L}(X)$ since $D(A)$ is dense in X and $\mathcal{R}_\alpha(\cdot)$ is bounded on $[0, 1]$ by (i).

Proof of (iv). For $x \in D$, proceeding as in the proof of (iii), we have

$$A\mathcal{R}_\alpha(t)x - Ax = \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t}\lambda^{-1} A F_\alpha(\lambda)(A + \widehat{B}(\lambda))x \, d\lambda.$$

Using now that $(A + \widehat{B}(\lambda))x \in D(A)$, the inequality and the assumption **(H3)** and proceeding as in the proof of (iii) we can conclude that $A\mathcal{R}_\alpha(t)x - Ax \to 0$ as $t \to 0$. The above remarks shows that $\|\mathcal{R}_\alpha(t)x - x\|_1 \to 0$ as $t \to 0$ for all $x \in D(A)$, since D is dense in $[D(A)]$ and $\mathcal{R}_\alpha(\cdot)$ is exponentially bounded in $\mathcal{L}([D(A)])$.

Theorem 2 *The operator funtion $t \to t^{\alpha-1}\mathcal{S}_\alpha(t)$ is exponentially bounded in $\mathcal{L}(X)$ and uniformly (strong) continuous on $(0, \infty)$.*

Comment: For $t \ge 1$, we have

$$\begin{aligned} \| t^{\alpha-1}\mathcal{S}_\alpha(t) \| &= \| \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t} F_\alpha(\lambda) d\lambda \| \\ &\le \left(\frac{C}{\pi}\int_r^\infty e^{st\cos\theta}\frac{ds}{s^\alpha} + \frac{C}{2\pi}\int_{-\theta}^{\theta} e^{tr\cos\xi} r^{1-\alpha} d\xi \right) \\ &\le \left(\frac{C}{\pi r^\alpha |\cos\theta|} + \frac{C\theta r^{1-\alpha}}{\pi} \right) e^{rt}. \end{aligned}$$

Since $F_\alpha(\cdot)$ is analytic on $\Sigma_{r,\theta}$, for $t \in (0, 1)$ we get

$$\begin{aligned} &\| t^{\alpha-1}\mathcal{S}_\alpha(t) \| \\ &= \| \frac{1}{2\pi i}\int_{\Gamma_{\frac{r}{t},\theta}} e^{\lambda t} F_\alpha(\lambda) d\lambda \| \\ &\le \frac{C}{\pi}\int_{\frac{r}{t}}^\infty e^{ts\cos\theta}\frac{ds}{s^\alpha} + \frac{C}{2\pi}\int_{-\theta}^{\theta} e^{r\cos\xi} d\xi \\ &\le \left(\frac{C}{\pi r^\alpha |\cos\theta|} + \frac{C\theta}{\pi} r^{1-\alpha} \right) e^{r}. \end{aligned}$$

This completes the proof of exponential boundedness.

For the uniform continuity, let $t > 0$ and $x \in X$, we have for $R > r$ and $s > 0$,

$$\| \frac{1}{2\pi i} \int_{\Gamma_{r,\theta} \cap \{\lambda \in \mathbb{C}: |\lambda| \geq R\}} e^{\lambda t} F_\alpha(\lambda) d\lambda \| \leq \frac{C}{\pi} \int_R^\infty e^{s\sigma \cos\theta} \frac{d\sigma}{\sigma^\alpha} \leq \frac{C e^{sR\cos(\theta)}}{\pi s R^\alpha \mid \cos(\theta) \mid}.$$

Therefore, for all $\epsilon > 0$, we can choose $R_t > r$ such that for all $s \in [\frac{t}{2}, \frac{3t}{2}]$ we have

$$\| \frac{1}{2\pi i} \int_{\Gamma_{r,\theta} \cap \{\lambda: |\lambda| \geq R_t\}} e^{\lambda t} F_\alpha(\lambda) d\lambda \| \leq \frac{\epsilon}{2}. \tag{14}$$

On the other hand, $e^{\lambda s} F_\alpha(\lambda) \to e^{\lambda t} F_\alpha(\lambda)$ as $s \to t$, uniformly on $\Gamma_{r,\theta} \cap \{\lambda \in \mathbb{C} :\mid \lambda \mid \leq R_t\}$, this implies, for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\| \int_{\Gamma_{r,\theta} \cap \{\lambda \in \mathbb{C}: |\lambda| \leq R\}} e^{\lambda s} F_\alpha(\lambda) d\lambda - \int_{\Gamma_{r,\theta} \cap \{\lambda \in \mathbb{C}: |\lambda| \leq R\}} e^{\lambda t} F_\alpha(\lambda) d\lambda \| < \frac{\epsilon}{2}. \tag{15}$$

By (14) and (15) we obtain for all $\epsilon > 0$, there exists $\delta > 0$ such that if $\mid t - s \mid < \delta$ we have

$$\| t^{\alpha-1} \mathcal{S}_\alpha(t) - s^{\alpha-1} \mathcal{S}_\alpha(s) \| < \epsilon.$$

This completes the prove.

Corollary 3 *Let $f \in L^1_{loc}(\mathbb{R}_+, X)$, then the convolution $t^{\alpha-1}\mathcal{S}_\alpha(t) * f(t) = \int_0^t (t-s)^{\alpha-1} \mathcal{S}_\alpha(t-s) f(s) ds$ exists (as a Bochner integral) and defines a continuous function from $\mathbb{R}_+$ into X.*

Lemma 4 *For every $\lambda \in \mathbb{C}$ with $Re(\lambda) > \max\{0, r\}$, $\widehat{\mathcal{R}_\alpha}(\lambda) = G_\alpha(\lambda)$ and $\widehat{(t^{\alpha-1}\mathcal{S}_\alpha)}(\lambda) = F_\alpha(\lambda)$.*

Comment:

Proof Using that $G_\alpha(\cdot)$ is analytic on $\Sigma_{r,\theta}$, and that the integrals involved in the calculus are absolutely convergent, we have

$$\begin{aligned} \widehat{\mathcal{R}_\alpha}(\lambda) = \int_0^\infty e^{-\lambda t} \mathcal{R}_\alpha(t) dt &= \int_0^\infty \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} e^{-(\lambda-\gamma)t} G_\alpha(\gamma)\, d\gamma dt \\ &= \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} (\lambda-\gamma)^{-1} G_\alpha(\gamma)\, d\gamma. \end{aligned}$$

By

$$\begin{aligned} \| \int_{C_{L,\theta}} (\lambda-\gamma)^{-1} G_\alpha(\gamma)\, d\gamma \| &\leq \int_{-\theta}^{\theta} \frac{C}{\mid \lambda - \gamma \mid\mid \gamma \mid} L d\xi \leq \int_{-\theta}^{\theta} \frac{C}{(L - \mid \lambda \mid) L} L d\xi \\ &= \frac{2\theta C}{(L - \mid \lambda \mid)}, \end{aligned}$$

we have $\int_{C_{L,\theta}} (\lambda-\gamma)^{-1} G_\alpha(\gamma)\, d\gamma$ converges to 0 as $L \to \infty$. Therefore

$$\begin{aligned} \widehat{\mathcal{R}_\alpha}(\lambda) &= \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} (\lambda-\gamma)^{-1} G_\alpha(\gamma)\, d\gamma \\ &= \lim_{L \to \infty} \left(\frac{1}{2\pi i} \int_{\{\Gamma_{r,\theta}:\, r \leq s \leq L\} \cup C_{L,\theta}} (\lambda-\gamma)^{-1} G_\alpha(\gamma)\, d\gamma \right) = G_\alpha(\lambda). \end{aligned}$$

From $F_\alpha(\cdot)$ is analytic on $\Sigma_{r,\theta}$ using the same argument as before we have

$$\begin{aligned}\widehat{t^{\alpha-1}\mathcal{S}_\alpha}(\lambda)=\int_0^\infty e^{-\lambda t}t^{1-\alpha}\mathcal{S}_\alpha(t)dt &= \int_0^\infty \frac{1}{2\pi i}\int_{\Gamma_{r,\,\theta}} e^{-(\lambda-\gamma)t}F_\alpha(\gamma)\,d\gamma dt\\ &= \frac{1}{2\pi i}\int_{\Gamma_{r,\,\theta}}(\lambda-\gamma)^{-1}F_\alpha(\gamma)\,d\gamma.\end{aligned}$$

Since

$$\begin{aligned}\| \int_{C_{L,\,\theta}}(\lambda-\gamma)^{-1}F_\alpha(\gamma)\,d\gamma \| &\leq \int_{-\theta}^{\theta}\frac{C}{\mid\lambda-\gamma\mid\mid\gamma\mid^\alpha}Ld\xi \leq \int_{-\theta}^{\theta}\frac{C}{(L-\mid\lambda\mid)L^\alpha}Ld\xi\\ &= \frac{2\theta CL}{(L-\mid\lambda\mid)L^\alpha},\end{aligned}$$

we have $\int_{C_{L,\,\theta}}(\lambda-\gamma)^{-1}F_\alpha(\gamma)\,d\gamma$ converges to 0 as $L\to\infty$. We infer

$$\begin{aligned}\widehat{t^{\alpha-1}\mathcal{S}_\alpha}(\lambda) &= \frac{1}{2\pi i}\int_{\Gamma_{r,\,\theta}}(\lambda-\gamma)^{-1}F_\alpha(\gamma)\,d\gamma\\ &= \lim_{L\to\infty}\left(\frac{1}{2\pi i}\int_{\{\Gamma_{r,\,\theta}:\ r\leq s\leq L\}\cup C_{L,\,\theta}}(\lambda-\gamma)^{-1}F_\alpha(\gamma)\,d\gamma\right)=F_\alpha(\lambda).\end{aligned}$$

Theorem 5 *The function $\mathcal{R}_\alpha(\cdot)$ is a α-resolvent operator for the system* (3)-(4).

Comment: Let $x\in D(A)$. From Lemma 4, for $Re(\lambda)>\max\{0,r\}$,

$$\widehat{\mathcal{R}_\alpha}(\lambda)[\lambda^{1-\alpha}(\lambda^\alpha I-A-\widehat{B}(\lambda))]x=x,$$

which implies

$$\lambda\widehat{\mathcal{R}_\alpha}(\lambda)x-x=\lambda^{1-\alpha}\widehat{\mathcal{R}_\alpha}(\lambda)Ax+\lambda^{1-\alpha}\widehat{\mathcal{R}_\alpha}(\lambda)\widehat{B}(\lambda)x,$$

we get

$$\lambda^\alpha\widehat{\mathcal{R}_\alpha}(\lambda)x-\lambda^{\alpha-1}x=\widehat{\mathcal{R}_\alpha}(\lambda)Ax+\widehat{\mathcal{R}_\alpha}(\lambda)\widehat{B}(\lambda)x,$$

and applying (8) and (WOLFGANG et al., 2002, Proposition 1.6.4) we obtain

$$\widehat{D_t^\alpha\mathcal{R}_\alpha}(\lambda)x=\widehat{\mathcal{R}_\alpha}(\lambda)Ax+\widehat{(\mathcal{R}_\alpha * B)}(\lambda)x.$$

By the uniqueness of the Laplace transform we get

$$D_t^\alpha\mathcal{R}_\alpha(t)x=\mathcal{R}_\alpha(t)Ax+\int_0^t\mathcal{R}_\alpha(t-s)B(s)xds.$$

Arguing as above but using the equality $[\lambda^{1-\alpha}(\lambda^\alpha I-A-\widehat{B}(\lambda))]\widehat{\mathcal{R}_\alpha}(\lambda)x=x$, we obtain that (9) holds. The proof is now completed.

We shall prove a result the existence of an analytic extension of resolvent operator.

Theorem 6 *The function $\mathcal{R}_\alpha:(0,\infty)\to\mathcal{L}(X)$ has an analytic extension to $\Sigma_{\delta,0}$, $\delta=\min\{\phi-\frac{\pi}{2},\pi-\phi\}$ and*

$$\mathcal{R}'_\alpha(z)=\frac{1}{2\pi i}\int_{\Gamma_{r,\theta}}\lambda e^{\lambda z}G_\alpha(\lambda)\,d\lambda,\qquad z\in\Sigma_{\delta,0}. \tag{16}$$

Comment: For $\lambda \in \Gamma_{r,\theta}$ and $z \in \Sigma_{\delta,0}$, we can write $\lambda z = s \mid z \mid e^{i(\arg(z)+\xi)}$ where $\frac{\pi}{2} < \arg(z) + \xi < \pi, -\theta \leq \xi \leq \theta$ and $s \geq r$. If $\mid z \mid > 1$, from (11) we get

$$\begin{aligned} \| \mathcal{R}_\alpha(z) \| &= \| \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} e^{\lambda z} G_\alpha(\lambda) d\lambda \| \\ &\leq \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} e^{Re(\lambda z)} \frac{C}{\mid \lambda \mid} \mid d\lambda \mid \\ &\leq \frac{C}{\pi} \int_r^\infty e^{s|z|\cos(\arg(z)+\theta)} \frac{ds}{s} + \frac{C}{2\pi} \int_{-\theta}^{\theta} e^{r|z|\cos(\arg(z)+\xi)} d\xi \\ &\leq \left(\frac{C}{\pi r|\cos(\arg(z)+\theta)|} + \frac{C\theta}{\pi} \right) e^{r|z|}. \end{aligned}$$

On the other hand, using that $G_\alpha(\cdot)$ is analytic on $\Sigma_{r,\theta}$, for $0 <\mid z \mid< 1$ we obtain

$$\begin{aligned} \| \mathcal{R}_\alpha(z) \| &= \| \frac{1}{2\pi i} \int_{\Gamma_{\frac{r}{|z|},\theta}} e^{\lambda z} G_\alpha(\lambda) d\lambda \| \\ &\leq \frac{C}{\pi} \int_{\frac{r}{|z|}}^\infty e^{s|z|\cos(\arg(z)+\theta)} \frac{ds}{s} + \frac{C}{2\pi} \int_{-\theta}^{\theta} e^{r\cos(\arg(z)+\xi)} d\xi \\ &\leq \left(\frac{C}{\pi} \int_r^\infty e^{u\cos(\arg(z)+\theta)} \frac{du}{u} + \frac{C}{2\pi} \int_{-\theta}^{\theta} e^{r\cos(\arg(z)+\xi)} d\xi \right) \\ &\leq \left(\frac{C}{\pi r|\cos(\arg(z)+\theta)|} + \frac{C\theta}{\pi} \right) e^{r}. \end{aligned}$$

This property allows us to define the extension $\mathcal{R}_\alpha(z)$ by this integral.

Similarly, the integral on the right hand side of (16) is also absolutely convergent in $\mathcal{L}(X)$ and strongly continuous on X for $|\arg z| < \delta$, we observe for $\lambda \in \Gamma_{r,\theta}$

$$\| \frac{e^{\lambda(z+h)} - e^{\lambda z}}{h} G_\alpha(\lambda) - \lambda e^{\lambda z} G_\alpha(\lambda) \| \leq \mid \frac{e^{\lambda(z+h)} - e^{\lambda z}}{h} - \lambda e^{\lambda z} \mid \frac{C}{r} \to 0, \mid h \mid \to 0,$$

and

$$\| \frac{e^{\lambda(z+h)} - e^{\lambda z}}{h} G_\alpha(\lambda) - \lambda e^{\lambda z} G_\alpha(\lambda) \| \leq e^{Re(\lambda z)} \frac{C}{\mid \lambda \mid} = K(\lambda),$$

where $K(\cdot)$ is integrable for $\lambda \in \Gamma_{r,\theta}$. From the Lebesgue dominated convergence theorem which implies that $\mathcal{R}'_\alpha(z)$ verifies (16).

In the next result we show that existence of resolvent operator implies in the existence of solutions for problem (1)-(2).

Theorem 7 *Let $x_0 \in [D(A)]$ and define $u(t) = \mathcal{R}_\alpha(t)x_0$. Then*

$$u \in C([0,\infty), [D(A)]) \cap C^\alpha((0,\infty), X),$$

and is a solutions of (3)-(4).

Comment:

By Theorem 1 (iii) and Theorem 6 it is easy to see that $u(t) = \mathcal{R}_\alpha(t)x_0$ is a function in $C([0,\infty), [D(A)]) \cap C^\alpha((0,\infty), X)$. By Theorem 5 we have $u(t) = \mathcal{R}_\alpha(t)x_0$ satisfies the problem (1)-(2).

3 Maximal mild solutions

In this section we study the concept continuation of local mild solutions and existence of global mild solution to (1)-(2). First, we study the existence of local the mild solution for the problem (1)-(2).

We denote by

- $C_\xi = \sup_{s\in[0,\xi]} \| f(s, u_0) \|$
- $D_\xi = \sup_{s\in[0,\xi]} \| f(s, u(s)) \|$
- $M_\xi = \max\left\{\sup_{s\in[0,\xi]} \| \mathcal{S}_\alpha(t) \|, \sup_{s\in[0,\xi]} \| \mathcal{P}_\alpha(t) \|\right\}$.

Now we define a concept of mild solution for the semilinear integro-differential fractional problem

$$\begin{aligned} D_t^\alpha u(t) &= Au(t) + \int_0^t B(t-s)u(s)ds + f(t, u(t)), \ t \in [0, a), \\ u(0) &= u_0, \end{aligned}$$

where $\alpha \in (0, 1)$ and f is a apropriate function. In the sequel, $\mathcal{R}_\alpha(\cdot)$ and $\mathcal{S}_\alpha(\cdot)$ is the α-resolvent operators and auxiliary resolvent operator studied in defined by (11) and (12) respectively.

Now we wil construct a notion of mild solution of the problem (1)-(2). Let $u : [0, \infty) \to X$ is a continuous functions satisfying (1)-(2). Then applying J_t^α at both sides of the equation (1) we have

$$\begin{aligned} u(t) &= u(0) + J_t^\alpha Au(t) + J_t^\alpha(B(t) * u(t)) + J_t^\alpha f(t, u(t)) \qquad (17) \\ &= u(0) + g_\alpha * Au(t) + g_\alpha * (B(t) * u(t)) + g_\alpha * f(t, u(t)). \end{aligned}$$

Now assuming that this function is of exponential type and is locally integrable, we apply that Laplace transform os both sides we obtain

$$\widehat{u}(\lambda) = \frac{u_0}{\lambda} + \frac{A\widehat{u}(\lambda)}{\lambda^\alpha} + \frac{\widehat{B}(\lambda)\widehat{u}(\lambda)}{\lambda^\alpha} + \frac{\widehat{f(u)}(\lambda)}{\lambda^\alpha},$$

where $\widehat{f(u)}(\lambda)$ is a Laplace transform of $f(t, u(t))$. We infer

$$\begin{aligned} \widehat{u}(\lambda) &= \lambda^{\alpha-1}(\lambda^\alpha I - A - \widehat{B}(\lambda))^{-1}u_0 + (\lambda^\alpha I - A - \widehat{B}(\lambda))^{-1}\widehat{f(u)}(\lambda) \\ &= G_\alpha(\lambda)u_0 + F_\alpha(\lambda)\widehat{f(u)}(\lambda) \\ &= \widehat{\mathcal{R}_\alpha}(t)u_0 + \widehat{t^{\alpha-1}\mathcal{S}_\alpha}(t)\widehat{f(u)}(\lambda) \\ &= \widehat{\mathcal{R}_\alpha}(t)u_0 + \widehat{t^{\alpha-1}\mathcal{S}_\alpha(t) * f(t, u(t))}. \end{aligned}$$

Finally applying the inverse of Laplace transform we end with the formula

$$u(t) = \mathcal{R}_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1}\mathcal{S}_\alpha(t-s)f(s, u(s))ds,$$

this equation inspires the next definitions.

Definition 3 *Let $\tau > 0$, a function $u : [0, \tau] \to X$ is called mild solution of (1)-(2) in $[0, \tau]$ if $u \in C([0, \tau], X)$ and*

$$u(t) = \mathcal{R}_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1}\mathcal{S}_\alpha(t-s)f(s, u(s))ds, \qquad (18)$$

holds for all $t \in [0, \tau]$.

Theorem 8 *Lef $f : [0,\infty) \times X \to X$ be a continuous function and locally Lipschitz in the second variable and uniformly with respect the first variable, that is, for each $x \in X$, there exists an open ball $B(x, R)$ and constant $L = L(B(x, R)) \geq 0$ such that*

$$\| f(t, y) - f(t, v) \| \leq L \| y - v \|,$$

for all $y, v \in B(x, R)$ and $t \in [0, \infty)$. Then, there exists $\tau_0 > 0$ such that (1)-(2) has a unique mild solutions in $[0, \tau_0]$.

Proof: Given $u_0 \in X$, let $B(u_0, r)$ and $L = L(B(u_0, r))$ be the Lipschitz constant of f. Given $b \in (0, r)$ fixed, by Theorem 1 and Theorem 2 we can choose $\tau_0 > 0$ such that

- $\| \mathcal{R}_\alpha(t)u_0 - u_0 \| \leq \frac{b}{2}$,
- and $\frac{M_{\tau_0}}{\alpha}(Lb + C_{\tau_0})t^\alpha \leq \frac{b}{2}$, for all $t \in [0, \tau_0]$.

We define

$$S(\tau_0) = \{u \in C([0, \tau_0], X) : u(0) = u_0 \text{ and } \| u(t) - u_0 \| \leq b \text{ for all t } \in [0, \tau_0]\}$$

with the norm $\| u \| = \sup_{t\in[0,\tau_0]} \| u(t) \|$ and the operator T on $S(\tau_0)$ by

$$T(u(t)) = \mathcal{R}_\alpha(t)u_0 + \int_0^t (t - s)^{\alpha-1}\mathcal{S}_\alpha(t - s)f(s, u(s))ds.$$

If $u \in S(\tau_0)$, we have $T(u(0)) = u_0$ and $T(u(t)) \in C([0, \tau_0], X)$. On the other hand, we have that

$$\begin{aligned}
&\| T(u(t)) - u_0 \| \\
&\quad \leq \| \mathcal{R}_\alpha(t)u_0 - u_0 \| \\
&\qquad + \int_0^t (t - s)^{\alpha-1} \| \mathcal{S}_\alpha(t - s) \| (\| f(s, u(s)) - f(s, u_0) \| + \| f(s, u_0) \|)ds \\
&\quad \leq \| \mathcal{R}_\alpha(t)u_0 - u_0 \| + \int_0^t (t - s)^{\alpha-1}M_{\tau_0}L \| u(s) - u_0 \| ds + \int_0^t (t - s)^{\alpha-1}M_{\tau_0}C_{\tau_0}ds \\
&\quad \leq \| \mathcal{R}_\alpha(t)u_0 - u_0 \| + M_{\tau_0}Lb\frac{t^\alpha}{\alpha} + M_{\tau_0}C_{\tau_0}\frac{t^\alpha}{\alpha} \\
&\quad \leq \| \mathcal{R}_\alpha(t)u_0 - u_0 \| + \frac{M_{\tau_0}}{\alpha}(Lb + C_{\tau_0})t^\alpha \leq \frac{b}{2} + \frac{b}{2} = b,
\end{aligned}$$

for all $t \in [0, \tau_0]$, this show that $TS(\tau_0) \subset S(\tau_0)$. If $u, v \in S(\tau_0)$ we obtain

$$\begin{aligned}
\| T(u(t)) - T(v(t)) \| &\leq \int_0^t (t - s)^{\alpha-1} \| \mathcal{S}_\alpha(t - s) \| \| f(s, u(s)) - f(s, v(s)) \| ds \\
&\leq \int_0^t (t - s)^{\alpha-1}M_{\tau_0}L \| u(s) - v(s) \| ds \\
&\leq \frac{M_{\tau_0}L\tau^\alpha}{\alpha} \sup_{s\in[0,\tau_0]} \| u(s) - v(s) \| .
\end{aligned}$$

This implies,

$$\parallel T(u) - T(v) \parallel \quad \leq \quad \frac{NL\tau^{\alpha}}{\alpha} \parallel u - v \parallel .$$

From $\frac{M_{\tau_0} L\tau^{\alpha}}{\alpha} \leq \frac{1}{2}$ by the Banach contraction principle we have that T has a unique fixed point in $S(\tau_0)$. This prove that (1)-(2) has a unique mild solutions in $[0, \tau_0]$.

Definition 4 *Let $u : [0, \tau] \to X$ be the unique local mild solution of (1)-(2) in $[0, \tau]$. If there exist $\tau^* > \tau$ and $u^* : [0, \tau^*] \to X$ is a local mild solution of (1)-(2) in $[0, \tau^*]$, the we say u^* is a continuation of u over $[0, \tau^*]$.*

Definition 5 *If $u : [0, \tau^*) \to X$ is the unique local mild solution of (1)-(2) in $[0, t]$ for all $t \in (0, \tau^*)$ and does not have a continuation, then we call it a maximal mild solution of (1)-(2).*

Theorem 9 *Lef $f : [0, \infty) \times X \to X$ be a continuous function and locally Lipschitz in the second variable and uniformly with respect the first variable, that is, for each $x \in X$, there exists an open ball $B(x, R)$ and constant $L = L(B(x, R)) \geq 0$ such that*

$$\parallel f(t, y) - f(t, v) \parallel \leq L \parallel y - v \parallel,$$

for all $y, v \in B(x, R)$ and $t \in [0, \infty)$. If $u : [0, \tau_0] \to X$ is a unique mild solution of (1)-(2) in $[0, \tau_0]$, then there exists a unique continuation solution u^* of u in some interval $[0, \tau_0 + \tau]$ with $\tau > 0$.

Proof: How $u(\tau_0) \in X$, let $B(u(\tau_0), r)$ and $L = L(B(u(\tau_0), r))$ be the Lipschitz constant of f. Given $b \in (0, r)$ fixed, by Theorem 1 and Theorem 2 we can choose $\tau > 0$ such that

- $\parallel \mathcal{R}_{\alpha}(t)u_0 - \mathcal{R}_{\alpha}(\tau_0)u_0 \parallel \leq \frac{b}{4}$,
- $\frac{M_{\tau_0+\tau}}{\alpha}(Lb + C_{\tau_0+\tau})t^{\alpha} \leq \frac{b}{4}$,
- $\frac{M_{\tau_0} D_{\tau_0}}{\alpha}[t^{\alpha} - (t - \tau_0)^{\alpha} - \tau_0^{\alpha}] \leq \frac{b}{4}$
- $\displaystyle\int_0^{\tau_0} (\tau_0 - s)^{\alpha-1} \parallel [\mathcal{S}_{\alpha}(t - s) - \mathcal{S}_{\alpha}(\tau_0 - s)] f(s, u(s)) \parallel ds \leq \frac{b}{4}$,

for all $t \in [\tau_0, \tau_0 + \tau]$.

We define

$$S(\tau_0 + \tau) = \{w \in C([0, \tau_0 + \tau], X) : w(t) = u(t) \text{ for all } t \in [0, \tau_0] \tag{19}$$

$$\text{and } \parallel w(t) - u(\tau_0) \parallel \leq b \text{ for all t } \in [\tau_0, \tau_0 + \tau]\} \tag{20}$$

with the norm $\parallel w \parallel = \sup_{t \in [0, \tau_0+\tau]} \parallel w(t) \parallel$ and the operator T on $S(\tau_0 + \tau)$ by

$$T(w(t)) = \mathcal{R}_{\alpha}(t)u_0 + \int_0^t (t - s)^{\alpha-1} \mathcal{S}_{\alpha}(t - s) f(s, w(s)) ds.$$

If $w \in S(\tau_0 + \tau)$, we have for all $t \in [0, \tau_0]$ that

$$T(w(t)) = \mathcal{R}_{\alpha}(t)u_0 + \int_0^t (t - s)^{\alpha-1} \mathcal{S}_{\alpha}(t - s) f(s, u(s)) ds = u(t),$$

$T(w(t)) \in C([0, \tau_0 + \tau], X)$ and

$$
\begin{aligned}
&\| T(w(t)) - u(\tau_0) \| \\
&\quad \leq \| \mathcal{R}_\alpha(t)u_0 - \mathcal{R}_\alpha(\tau_0)u_0 \| \\
&\quad\quad + \int_0^{\tau_0} \| [(t-s)^{\alpha-1}\mathcal{S}_\alpha(t-s) - (\tau_0 - s)^{\alpha-1}\mathcal{S}_\alpha(\tau_0 - s)] f(s, w(s)) \| \, ds \\
&\quad\quad + \int_{\tau_0}^{t} \| (t-s)^{\alpha-1}\mathcal{S}_\alpha(t-s) f(s, w(s)) \| \, ds \\
&\quad \leq \| \mathcal{R}_\alpha(t)u_0 - \mathcal{R}_\alpha(\tau_0)u_0 \| \\
&\quad\quad + \int_0^{\tau_0} [(t-s)^{\alpha-1} - (\tau_0 - s)^{\alpha-1}] \| \mathcal{S}_\alpha(t-s) f(s, u(s)) \| \, ds \\
&\quad\quad + \int_0^{\tau_0} (\tau_0 - s)^{\alpha-1} \| [\mathcal{S}_\alpha(t-s) - \mathcal{S}_\alpha(\tau_0 - s)] f(s, u(s)) \| \, ds \\
&\quad\quad + \int_{\tau_0}^{t} \| (t-s)^{\alpha-1}\mathcal{S}_\alpha(t-s)[f(s, w(s)) - f(s, u(\tau_0)) \| \, ds \\
&\quad\quad + \int_{\tau_0}^{t} \| (t-s)^{\alpha-1}\mathcal{S}_\alpha(t-s) f(s, u(\tau_0)) \| \, ds \\
&\quad \leq \frac{b}{4} + \frac{M_{\tau_0} D_{\tau_0}}{\alpha}[t^\alpha - (t-\tau_0)^\alpha - \tau_0^\alpha] + \frac{b}{4} \\
&\quad\quad + \frac{M_{\tau_0+\tau}}{\alpha} L \| w(t) - u(\tau_0) \| t^\alpha + \frac{M_{\tau_0+\tau} C_{\tau_0+\tau}}{\alpha} t^\alpha \leq b.
\end{aligned}
$$

for all $t \in [\tau_0, \tau_0 + \tau]$, this show that $TS(\tau_0 + \tau) \subset S(\tau_0 + \tau)$.
If $u, v \in S(\tau_0 + \tau)$ we obtain

$$
\begin{aligned}
\| T(u(t)) - T(v(t)) \| &\leq \int_0^t (t-s)^{\alpha-1} \| \mathcal{S}_\alpha(t-s) \| \| f(s, u(s)) - f(s, v(s)) \| \, ds \\
&\leq \int_0^t (t-s)^{\alpha-1} M_{\tau_0+\tau} L \| u(s) - v(s) \| \, ds \\
&\leq \frac{M_{\tau_0+\tau} L \tau^\alpha}{\alpha} \sup_{s \in [0, \tau_0+\tau]} \| u(s) - v(s) \| .
\end{aligned}
$$

This implies,

$$
\| T(u) - T(v) \| \leq \frac{M_{\tau_0+\tau} L \tau^\alpha}{\alpha} \| u - v \| .
$$

From $\frac{M_{\tau_0+\tau} L \tau^\alpha}{\alpha} \leq \frac{1}{4}$ by the Banach contraction principle we have that T has a unique fixed point in $S(\tau_0 + \tau)$. This prove that (1)-(2) has a unique mild solutions in $[0, \tau_0 + \tau]$. Therefore there exists a unique continuation solution of $u(\cdot)$.

Theorem 10 *Let $f : [0, \infty) \times X \to X$ be a continuous, locally Lipschitz in the second variable, uniformly with respect to the first variable, and bounded. Then the problem (1)-(2) has a global mild solution in $[0, \infty)$ or exist $\omega \in (0, \infty)$ such that $u : [0, \omega) \to X$ is a maximal mild solution of (1)-(2), and* $\limsup_{t \to \omega^-} \| u(t) \| = \infty$.

Proof: Let

$$H := \{\tau \in [0,\infty) : \exists u_\tau : [0,\tau] \to X \text{ unique mild solution to (1)-(2) in } [0,\tau]\}.$$

We denote by $w = \sup H$, we can consider a continuous function $u : [0,w) \to X$ that is a mild solution of (1)-(2) in $[0,w)$. If $w = \infty$, the u is a global mild solution in $[0,\infty)$. By the other side, if $w < \infty$ we will show that $\limsup_{t\to w} \| u(t) \| = \infty$. By contradiction, suppose that there exists $K < \infty$ such that $\| u(t) \| \leq K$ for all $t \in [0,w)$. Let $\{t_n\} \subset [0,w)$ is a sequence that converges to w. For all $\epsilon > 0$, there exist $N \in \mathbb{N}$ and $\gamma \in (0,w)$, such that, if $m,n > N$, we get

- $\| \mathcal{R}_\alpha(t_n)u_0 - \mathcal{R}_\alpha(t_m)u_0 \| \leq \frac{\epsilon}{7}$,
- $\frac{M_w D_w}{\alpha} \mid t_n - t_m \mid \leq \frac{\epsilon}{7}$,
- $\frac{M_w D_w}{\alpha} \mid t^\alpha - (t-\tau_0)^\alpha - \tau_0^\alpha \mid \leq \frac{\epsilon}{7}$
- $\frac{2M_w D_w}{\alpha}(w-\gamma) \leq \frac{\epsilon}{5}$
- $t_n > \gamma$ and $t_m > \gamma$
- $\displaystyle\int_0^\gamma (t_n - s)^{\alpha-1} \| [\mathcal{S}_\alpha(t_n - s) - \mathcal{S}_\alpha(w-s)]f(s,u(s)) \| \, ds \leq \frac{\epsilon}{7}$,
- $\displaystyle\int_0^\gamma (t_m - s)^{\alpha-1} \| [\mathcal{S}_\alpha(t_m - s) - \mathcal{S}_\alpha(w-s)]f(s,u(s)) \| \, ds \leq \frac{\epsilon}{7}$.

Without loss of generality, than $t_n > t_m$, it follows from the estimative

$$\begin{aligned}
\| u(t_n) - u(t_m) \| &\leq \| \mathcal{R}_\alpha(t_n)u_0 - \mathcal{R}_\alpha(t_m)u_0 \| \\
&+ \int_0^{t_m} \| (t_n - s)^{\alpha-1}\mathcal{S}_\alpha(t_n - s) - (t_m - s)^{\alpha-1}\mathcal{S}_\alpha(t_m - s)]f(s,u(s)) \| \, ds \\
&+ \int_{t_m}^{t_n} \| (t_n - s)^{\alpha-1}\mathcal{S}_\alpha(t - s)f(s,u(s)) \| \, ds \\
&\leq \| \mathcal{R}_\alpha(t_n)u_0 - \mathcal{R}_\alpha(t_m)u_0 \| \\
&+ \int_0^{t_m} \| (t_n - s)^{\alpha-1}[\mathcal{S}_\alpha(t_n - s) - \mathcal{S}_\alpha(w - s)]f(s,u(s)) \| \, ds \\
&+ \int_0^{t_m} \| [(t_n - s)^{\alpha-1}\mathcal{S}_\alpha(w - s) - (t_m - s)\mathcal{S}_\alpha(w - s)]f(s,u(s)) \| \, ds \\
&+ \int_0^{t_m} \| (t_m - s)^{\alpha-1}[\mathcal{S}_\alpha(t_m - s) - \mathcal{S}_\alpha(w - s)]f(s,u(s)) \| \, ds \\
&+ \int_{t_m}^{t_n} \| (t_n - s)^{\alpha-1}\mathcal{S}_\alpha(t - s)f(s,u(s)) \| \, ds \\
&= \| \mathcal{R}_\alpha(t_n)u_0 - \mathcal{R}_\alpha(t_m)u_0 \| + I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

We observe

$$\begin{aligned}
I_3 &= \int_0^{t_m} \| (t_m - s)^{\alpha-1}[\mathcal{S}_\alpha(t_m - s) - \mathcal{S}_\alpha(w - s)] f(s, u(s)) \| \, ds \\
&\leq \int_0^{\gamma} \| (t_m - s)^{\alpha-1}[\mathcal{S}_\alpha(t_m - s) - \mathcal{S}_\alpha(w - s)] f(s, u(s)) \| \, ds \\
&+ \int_\gamma^{t_m} \| (t_m - s)^{\alpha-1}[\mathcal{S}_\alpha(t_m - s) - \mathcal{S}_\alpha(w - s)] f(s, u(s)) \| \, ds \\
&\leq \frac{\epsilon}{7} + 2M_w D_w \int_\gamma^{t_m} (t_m - s)^{\alpha-1} ds \leq \frac{\epsilon}{7} + \frac{2M_w D_w}{\alpha}(t_m - \gamma)^\alpha \\
&\leq \frac{\epsilon}{7} + \frac{\epsilon}{7}.
\end{aligned}$$

By the same way, we can show

$$I_1 \leq \int_0^{t_n} \| (t_n - s)^{\alpha-1}[\mathcal{S}_\alpha(t_n - s) - \mathcal{S}_\alpha(w - s)] f(s, u(s)) \| \, ds \leq \frac{\epsilon}{7} + \frac{\epsilon}{7}.$$

We get

$$\begin{aligned}
I_2 &= \int_0^{t_m} \| [(t_n - s)^{\alpha-1} \mathcal{S}_\alpha(w - s) - (t_m - s)\mathcal{S}_\alpha(w - s)] f(s, u(s)) \| \, ds \\
&\leq M_w D_w \int_0^{t_m} [(t_n - s)^{\alpha-1} \mathcal{S}_\alpha(w - s) - (t_m - s)] ds \\
&\leq \frac{M_w D_w}{\alpha}[(t_n - t_m)^\alpha - t_n^\alpha - t_m^\alpha] \leq \frac{\epsilon}{7}.
\end{aligned}$$

and

$$\begin{aligned}
I_4 &= \int_{t_m}^{t_n} \| (t_n - s)^{\alpha-1} \mathcal{S}_\alpha(t - s) f(s, u(s)) \| \, ds \\
&\leq M_w D_w \frac{(t_n - t_m)^\alpha}{\alpha} \leq \frac{\epsilon}{7}.
\end{aligned}$$

Therefore

$$\| u(t_n) - u(t_m) \| < \epsilon.$$

This shows that $\{u(t_n)\}$ is a Cauchy sequence and therefore it has a limit, $u_w \in X$. Then, we may extende u over $[0, w]$, obtaining the equality

$$u(t) = \mathcal{R}_\alpha(t) u_0 + \int_0^t (t - s)^{\alpha-1} \mathcal{S}_\alpha(t - s) f(s, u(s)) ds,$$

for all $t \in [0, w]$, by Theorem 9, we can extend the solution to some bigger interval, wich is a contradiction with the definition of w, by the contradiction above,

$$\limsup_{t \to \omega^-} \| u(t) \| = \infty.$$

This finished the proof.

4 References

AGARWAL, R. P.; SANTOS, J. P. C. dos; CUEVAS, C. Analytic resolvent operator and existence results for fractional integro-differential equations. **Journal of abstract differential equations and applications**, v. 2, n. 2, p. 26–47, 2012.

ANDRADE, B. et al. Semilinear fractional differential equations: global solutions, critical nonlinearities and comparison results. **Topological Methods in Nonlinear Analysis**, Nicolaus Copernicus University, Juliusz P. Schauder Centre for Nonlinear Studies, v. 45, n. 2, p. 439–467, 2015.

BAJLEKOVA, E. G. et al. **Fractional evolution equations in Banach spaces**. 2001. 117 f. Tese (Doutorado) — Technische Universiteit Eindhoven, Eindhoven, 2001.

CARVALHO NETO, P. M. **Fractional differential equations**: a novel study of local and global solutions in banach spaces. 2013. 124 f. Tese (Doutorado em Matemática) — Universidade de São Paulo, São Carlos, 2013.

DA PRATO, G.; IANNELLI, M. Existence and regularity for a class of integrodifferential equations of parabolic type. **Journal of Mathematical Analysis and Applications**, Orlando, v. 112, n. 1, p. 36–55, 1985.

DA PRATO, G.; LUNARDI, A. Solvability on the real line of a class of linear volterra integrodifferential equations of parabolic type. **Annali di matematica pura ed applicata**, Heidelberg, v. 150, n. 1, p. 67–117, 1988.

EL-SAYED, A. M.; HERZALLAH, M. A. Continuation and maximal regularity of an arbitrary (fractional) order evolutionary integro-differential equation. **Applicable Analysis**, New York, v. 84, n. 11, p. 1151–1164, 2005.

GRIMMER, R.; KAPPEL, F. Series expansions for resolvents of volterra integrodifferential equations in banach space. **SIAM Journal on Mathematical Analysis**, Philadelphia, v. 15, n. 3, p. 595–604, 1984.

GRIMMER, R.; PRITCHARD, A. J. Analytic resolvent operators for integral equations in banach space. **Journal of Differential equations**, New York, v. 50, n. 2, p. 234–259, 1983.

GRIMMER, R.; PRÜSS, J. On linear volterra equations in banach spaces. **Computers & Mathematics with Applications**, Amsterdam, v. 11, n. 1-3, p. 189–205, 1985.

GRIPENBERG, G.; LONDEN, S.-O.; STAFFANS, O. **Volterra integral and functional equations**. [S.l.]: Cambridge University Press, 1990.

HERNÁNDEZ, E.; O'REGAN, D.; BALACHANDRAN, K. Existence results for abstract fractional differential equations with nonlocal conditions via resolvent operators. **Indagationes mathematicae**, Amsterdam, v. 24, n. 1, p. 68–82, 2013.

KEYANTUO, V.; LIZAMA, C.; WARMA, M. Asymptotic behavior of fractional order semilinear evolution equations. **Differential and Integral Equations**, Athens, v. 26, n. 7/8, p. 757–780, 2013.

KOSTIC, M. **Abstract Volterra integro-differential equations**. Boca Raton: CRC Press, 2015.

LI, Y.-N.; SUN, H.-R.; FENG, Z. Fractional abstract cauchy problem with order $\alpha \in (1, 2)$. **Dynamics of Partial Differential Equations**, Boston, v. 13, n. 2, p. 155–177, 2016.

LI, Y.-N.; SUN, H.-R.; FENG, Z. Fractional abstract cauchy problem with order $\alpha \in (1, 2)$. **Dynamics of Partial Differential Equations**, Boston, v. 13, n. 2, p. 155–177, 2016.

LUNARDI, A. Laplace transform methods in integrodifferential equations. **The Journal of Integral Equations**, New York, p. 185–211, 1985.

LUNARDI, A. On the linear heat equation with fading memory. **SIAM Journal on Mathematical analysis**, Philadelphia, v. 21, n. 5, p. 1213–1224, 1990.

PAZY, A. **Semigroups of linear operators and applications to partial differential equations**. [S.l.]: Springer Science & Business Media, 2012. v. 44.

PONCE, R. Bounded mild solutions to fractional integro-differential equations in banach spaces. **Semigroup Forum**, Berlin, v. 87, n. 2, p. 377–392, 2013.

PRÜSS, J. **Evolutionary integral equations and applications**. [S.l.]: Birkhäuser, 2013. v. 87.

SAMKO, S. G.; KILBAS, A. A.; MARICHEV, O. I. Fractional integrals and derivatives: theory and applications. **Gordon and Breach Science Publishers**, New York, 1993.

SANTOS, J. P. C. Fractional resolvent operator with $\alpha \in (0, 1)$ and applications. **Fractional Differential Calculus**, Zagreb, v. 9, n. 2, p. 187–208, 2019.

SANTOS, J. P. C.; HENRÍQUEZ, H.; HERNÁNDEZ, E. Existence results for neutral integro-differential equations with unbounded delay. **Journal of Integral Equations and Applications**, New York, p. 289–330, 2011.

SANTOS, J. P. C.; HENRÍQUEZ, H. R. Existence of s-asymptotically ω-periodic solutions to abstract integro-differential equations. **Applied Mathematics and Computation**, Amsterdam, v. 256, p. 109–118, 2015.

SFORZA, D. Parabolic integrodifferential equations with singular kernels. **The Journal of Integral Equations and Applications**, New York, v. 4, p. 601–623, 1991.

WANG, R.-N.; CHEN, D.-H.; XIAO, T.-J. Abstract fractional cauchy problems with almost sectorial operators. **Journal of Differential Equations**, Amsterdam, v. 252, n. 1, p. 202–235, 2012.

WOLFGANG, A. et al. Vector-valued laplace transforms and cauchy problems. **Birkhäuser Basel**, 2002.

ZHOU, Y.; JIAO, F. Nonlocal cauchy problem for fractional evolution equations. **Nonlinear analysis:** real world applications, Amsterdam, v. 11, n. 5, p. 4465–4475, 2010.