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Variational approach for ψ-Hilfer fractional operator

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Abstract

In this work, we aim to present properties that characterize a variational structure in the ψ-fractional space $\mathbb{H}_p^{\alpha,\beta;\psi}(\Omega)$ for the ψ-Hilfer fractional operator. The importance of these results are strongly used in the investigation of fractional differential equations involving problems such as: p-Laplacian, concave-convex, singular double phases, among others.

Keywords: ψ-Hilfer fractional operator. variational approach. ψ-fractional spaces.

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1 Introduction and motivation

The fractional calculus is well consolidated and well known, motivated by a question in 1695 through a letter between L'Hospital and Leibniz. What question could we ask today, so that in the future it would generate a great impact on the scientific community? A priory, there are open questions and problems that would strongly impact the area, i.e., a geometric interpretation for a fractional derivative. Over these 326 years, numerous fractional integrals and fractional derivatives were introduced, each with its own importance (SOUSA; OLIVEIRA, 2018, 2019b, 2019c; DIETHELM; FORD, 2002; KILBAS; SRIVASTAVA; TRUJILLO, 2006; LAKSHMIKANTHAM; VATSALA, 2008; SOUSA; FREDERICO; OLIVEIRA, 2020; SOUSA; MACHADO; OLIVEIRA, 2020). However, in mid 2018, Sousa and Oliveira (SOUSA; OLIVEIRA, 2018) raised the following question: What is the best derivative to be chosen to fit data? The question was asked due to the large number of existing fractional derivatives. In this sense, motivated by Caputo fractional derivative with respect to another function and by the Hilfer fractional derivative, Sousa and Oliveira, they introduced the so-called ψ-Hilfer fractional derivative, presented a wide class of particular cases and discussed important properties of fractional calculus, in particular, the Leibniz I and II type rules (SOUSA; OLIVEIRA, 2019b).

On the other hand, the theory of fractional differential equations followed the same steps as fractional calculus, especially in recent years in a growing exponential of great relevance in the scientific community (DIETHELM; FORD, 2002; KILBAS; SRIVASTAVA; TRUJILLO, 2006; LAKSHMIKANTHAM; VATSALA, 2008; NOROUZI; N'GUÉRÉKATA, 2021, 2021; SOUSA; BENCHOHRA; N'GUÉRÉKATA, 2020). The investigation of problems of fractional differential equations involving variational methods, in recent years, has drawn a lot of attention from researchers. However, the results presented so far are very restricted to the Caputo and Riemann-Liouville fractional derivatives. For a reading of results on fractional differential equations involving variational methods, see (GHANMI; ZHANG, 2019; ZHAO; TANG, 2017; JIAO; ZHOU, 2011; NYAMORADI; TAYYEBI, 2018; ZHANG; LI, 2015) and the references therein.

Let $\theta = (\theta_1, \theta_2, ..., \theta_N)$, $T = (T_1, T_2, ..., T_N)$ and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$ where $0 < \alpha_1, \alpha_2, ..., \alpha_N < 1$ with $\theta_j < T_j$, for all $j \in \{1, 2, ..., N\}$, $N \in \mathbb{N}$. Also put $\Omega = I_1 \times I_2 \times \cdots \times I_N = [\theta_1, T_1] \times [\theta_2, T_2] \times \cdots \times [\theta_N, T_N]$ where $T_1, T_2, ..., T_N$ and $\theta_1, \theta_2, ..., \theta_N$ positive constants. Consider also $\psi(\cdot)$ be an increasing and positive monotone function on $(\theta_1, T_1), (\theta_2, T_2), ..., (\theta_N, T_N)$, having a continuous derivative $\psi'(\cdot)$ on $(\theta_1, T_1], (\theta_2, T_2], ..., (\theta_N, T_N]$. The ψ-Riemann-Liouville fractional partial integral of order α of N-variables $\xi = (\xi_1, \xi_2, ..., \xi_N) \in L^1(\Omega)$ denoted by $\mathbf{I}^{\alpha,\psi}_{\theta,x_j}(\cdot)$, is defined by (SOUSA; OLIVEIRA, 2019a)

$$\mathbf{I}^{\alpha,\psi}_{\theta,x_j}\xi(x_j) = \frac{1}{\Gamma(\alpha_j)} \int\int \cdots \int_{\Omega} \psi'(s_j)(\psi(x_j) - \psi(s_j))^{\alpha_j - 1}\xi(s_j)ds_j$$

with $\psi'(s_j)(\psi(x_j) - \psi(s_j))^{\alpha_j-1} = \psi'(s_1)(\psi(x_1) - \psi(s_1))^{\alpha_1-1}\psi'(s_2)(\psi(x_2) - \psi(s_2))^{\alpha_2-1} \cdots \psi'(s_N)(\psi(x_N) - \psi(s_N))^{\alpha_N-1}$ where $\Gamma(\alpha_j) = \Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_N)$, $\xi(s_j) = \xi(s_1)\xi(s_2)\cdots\xi(s_N)$, $ds_j = ds_1 ds_2 \cdots ds_N$, for all $j \in \{1, 2, ..., N\}$. Analogously, it is defined $\mathbf{I}^{\alpha,\psi}_{T,x_j}(\cdot)$.

On the other hand, let $\xi, \psi \in C^n(\Omega)$ two functions such that ψ is increasing and $\psi'(x_j) \neq 0$ $j \in \{1, 2, ..., N\}$, $x_j \in \Omega$. The ψ-Hilfer fractional partial derivative of N-variables, denoted by ${}^{\mathbf{H}}\mathbf{D}^{\alpha,\beta;\psi}_{\theta,x_j}(\cdot)$, of order α and type β $(0 \leq \beta \leq 1)$, is defined by (SOUSA; OLIVEIRA, 2019a)

$${}^{\mathbf{H}}\mathbf{D}^{\alpha,\beta;\psi}_{\theta,x_j}\xi(x_j) = \mathbf{I}^{\beta(1-\alpha),\psi}_{\theta,x_j}\left(\frac{1}{\psi'(x_j)}\frac{\partial^N}{\partial x_j}\right)\mathbf{I}^{(1-\beta)(1-\alpha),\psi}_{\theta,x_j}\xi(x_j) \tag{1}$$

with $\partial x_j = \partial x_1, \partial x_2 \cdots \partial x_N$ and $\psi'(x_j) = \psi'(x_1)\psi'(x_2)\cdots\psi'(x_N)$, for all $j \in \{1, 2, ..., N\}$. Analogously it is defined ${}^{\mathbf{H}}\mathbf{D}^{\alpha,\beta;\psi}_{T,x_j}(\cdot)$. Furthermore, we have the ψ-Hilfer-Caputo fractional partial derivative of of N-variables, denoted by ${}^{\mathbf{H}}_{c}\mathbf{D}^{\alpha,\beta;\psi}_{\theta,x_j}(\cdot)$, of order α and type β ($0 \leq \beta \leq 1$), is defined by (SOUSA; OLIVEIRA, 2019a)

$$ {}^{\mathbf{H}}\mathbf{D}^{\alpha,\beta;\psi}_{T,x_j}\xi(x_j) = \mathbf{I}^{(1-\beta)(1-\alpha),\psi}_{T,x_j}\left(-\frac{1}{\psi'(x_j)}\frac{\partial^N}{\partial x_j}\right)\mathbf{I}^{\beta(1-\alpha),\psi}_{T,x_j}\xi(x_j) \tag{2} $$

Throughout this work, we will use the following notations $\mathbf{I}^{\alpha,\psi}_{T}(\cdot) := \mathbf{I}^{\alpha,\psi}_{T,x_j}(\cdot)$, $\mathbf{I}^{\alpha,\psi}_{\theta}(\cdot) := \mathbf{I}^{\alpha,\psi}_{\theta,x_j}(\cdot)$, ${}^{\mathbf{H}}\mathbf{D}^{\alpha,\beta;\psi}_{\theta}(\cdot) := {}^{\mathbf{H}}\mathbf{D}^{\alpha,\beta;\psi}_{\theta,x_j}(\cdot)$ and ${}^{\mathbf{H}}_{c}\mathbf{D}^{\alpha,\beta;\psi}_{T}(\cdot) := {}^{\mathbf{H}}_{c}\mathbf{D}^{\alpha,\beta;\psi}_{T,x_j}(\cdot)$.

Let $0 < \alpha < 1$, $p \in (1, \infty)$ and $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$. Further, denote by $L^q_{\psi}(\Omega)$ the space $L^q_{\mu,\psi}(\Omega)$, where $\mu(x) \equiv 1$. If $u \in L^q_{\psi}(\Omega)$ and $v \in L^p_{\psi}(\Omega)$, then the following integration by parts (SOUSA et al., 2022)

$$ \int_{\Omega}(I^{\alpha,\psi}_{a+}v(x))u(x)\psi'(x)\mathrm{d}x = \int_{\Omega}v(x)(I^{\alpha,\psi}_{b-}u(x))\psi'(x)\mathrm{d}x $$

holds.

Let $0 < \alpha < 1$, $\beta \in [0, 1]$, $u \cdot v \in AC(\Omega)$ and $-\frac{1}{\psi'(\cdot)}I^{\beta(1-\alpha),\psi}_{T-}u \in L^2_{\psi}(\Omega)$. Then, (SOUSA et al., 2022)

$$ \begin{aligned} \int_{\Omega}{}^{H}_{C}\mathbf{D}^{\alpha,\beta;\psi}_{T-}u(x)v(x)\psi'(x)\mathrm{d}x &= \lim_{x\to 0^+} I^{(1-\alpha)(1-\beta),\psi}_{0+}v(x)I^{\beta(1-\alpha),\psi}_{T}u(x) \\ &- \lim_{x\to T^-} I^{(1-\alpha)(1-\beta),\psi}_{0+}v(x)I^{\beta(1-\alpha),\psi}_{T-}u(x) + \int_0^T u(x)\,{}^{\mathbf{H}}\mathbf{D}^{\alpha,\beta;\psi}_{0+}v(x)\psi'(x)\mathrm{d}x. \end{aligned} $$

Motivated by the ψ-Hilfer fractional derivative and by variational methods, Sousa et al. (SOUSA; ZUO; O'REGAN, 2021). constructed a variational structure and the ψ-fractional space. In this sense, motivated by this work, numerous other works were investigated (SOUSA, 2021; SOUSA; PULIDO; OLIVEIRA, 2021; SOUSA et al., 2022; SOUSA; TAVARES; LEDESMA, 2021; SOUSA et al., 2022; SOUSA, 2022). In this present work, we aim to present some results involving the ψ-Hilfer fractional derivative that guarantee a variational structure and that allowed to discuss variational results of great importance, in particular, involving problems p-Laplacian, concave-convex, singular double phase. In the vast majority of results, Nehari manifold techniques, fibration application, Mountain step theorem, Ekeland's method, among other methods, are important tools.

2 Variational approach

Here we will present some results on the ψ-fractional space in order to build a variational structure.

The space of p-integrable functions with respect to a function ψ is defined as (SOUSA; ZUO; O'REGAN, 2021)

$$ L^p(\Omega) = \left\{u : \Omega \longrightarrow \mathbb{R} \int_{\Omega}|\xi(x)|^p\mathrm{d}x < +\infty\right\}, $$

with norm

$$ \|u\|_{L^p(\Omega)} = \left(\int_{\Omega}|\xi(x)|^p\right)^{\frac{1}{q}}. $$

Definition 1 (SOUSA; ZUO; O'REGAN, 2021) *Let* $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$ *and* $1 < p < \infty$. *The* ψ*-fractional derivative space* $\mathbb{H}_p^{\alpha,\beta;\psi} := \mathbb{H}_p^{\alpha,\beta;\psi}(\Omega,\mathbb{R})$ *is defined by the closure of* $C_0^{\infty}(\Omega,\mathbb{R})$, *and is given by*

$$\begin{aligned}\mathbb{H}_p^{\alpha,\beta;\psi} &= \left\{ \xi \in L^p(\Omega,\mathbb{R});\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}\xi \in L^p(\Omega,\mathbb{R}), \xi(x) = 0 \right\} \\ &= \overline{C_0^{\infty}(\Omega,\mathbb{R})} \end{aligned} \tag{3}$$

with the following norm

$$\|\xi\|_{\mathbb{H}_p^{\alpha,\beta;\psi}} = \left(\|\xi\|_{L_\psi^p}^p + \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}\xi \right\|_{L_\psi^p}^p \right)^{1/p}, \tag{4}$$

where ${}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}(\cdot)$ *is the* ψ*-Hilfer fractional derivative with* $0 < \alpha \leq 1$ *and* $0 \leq \beta \leq 1$.

Choosing $p = 2$, in definition Eq.(3), we have the ψ-fractional derivative space $\mathbb{H}_2^{\alpha,\beta;\psi}$ is defined on $\overline{C_0^{\infty}(\Omega,\mathbb{R})}$ with respect to the norm

$$\|\xi\|_{\mathbb{H}_2^{\alpha,\beta;\psi}} = \left(\int_\Omega |\xi(x)|^2\, dx + \int_\Omega \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}\xi(x) \right|^2 dx \right)^{1/2}.$$

The space $\mathbb{H}_2^{\alpha,t;\psi}$ is a Hilbert space with the norm

$$\|\xi\|_{\mathbb{H}_2^{\alpha,\beta;\psi}} = \left(\int_\Omega \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}\xi(x) \right|^2 dx \right)^{1/2}$$

with $0 < \alpha \leq 1$ and $0 \leq \beta \leq 1$.

Lemma 2 (SOUSA; ZUO; O'REGAN, 2021; SOUSA; TAVARES; LEDESMA, 2021; SOUSA et al., 2022) *Let* $0 < \alpha \leq 1$ *and* $1 \leq p < \infty$. *For any* $\xi \in L^p([0,T],\mathbb{R})$, *we have*

$$\left\| \mathbf{I}_{a+}^{\alpha;\psi} \right\|_{L^p[0,T]} \leq \frac{(\psi(T) - \psi(0))^\alpha}{\Gamma(\alpha+1)} \|\xi\|_{L^p[0,T]}$$

for any $t \in [0,T]$.

Proposition 3 (SOUSA; ZUO; O'REGAN, 2021; SOUSA; TAVARES; LEDESMA, 2021; SOUSA et al., 2022) *Let* $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$ *and* $1 < p < \infty$. *For all* $\xi \in \mathbb{H}_p^{\alpha,\beta;\psi}$, *if* $1 - \alpha \geq 1/p$ *or* $\alpha > 1/p$, *we have*

$$\|\xi\|_{L^p} \leq \frac{(\psi(T) - \psi(0))^\alpha}{\Gamma(\alpha+1)} \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}\xi \right\|_{L^p}. \tag{5}$$

Moreover, if $\alpha > 1/p$ *and* $\dfrac{1}{p} + \dfrac{1}{q} = 1$, *then*

$$\|\xi\|_\infty \leq \frac{(\psi(T) - \psi(0))^{\alpha - 1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}} \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}\xi \right\|_{L^p}, \tag{6}$$

where $\|\xi\|_\infty = \sup\limits_{t\in[0,T]} |\xi(t)|$.

From the inequality (6), *we also have*

$$\|\xi\|_\infty \leq \frac{(\psi(T) - \psi(0))^{\alpha - 1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}} \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}\xi \right\|_{\mathbb{H}_p^{\alpha,\beta;\psi}}$$

that $\mathbb{H}_p^{\alpha,\beta;\psi}$ *is continuously injected into* $C([0,T])$ *for* $\alpha > \frac{1}{p}$.

According to (5), we can consider $\mathbb{H}_p^{\alpha,\beta;\psi}$ with respect to the equivalent norm

$$\|\xi\| = \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi \right\|_{L^p}. \tag{7}$$

Proposition 4 (SOUSA; ZUO; O'REGAN, 2021; SOUSA; TAVARES; LEDESMA, 2021; SOUSA et al., 2022) *Let $0 < \alpha < 1$ and $0 \leq \beta \leq 1$. For any $\xi \in \mathbb{H}_p^{\alpha,\beta;\psi}$, we have*

$$\left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi \right\|_{L^p} \leq \frac{(\psi(T) - \psi(0))^{\alpha}}{\Gamma(\alpha+1)} \left\| {}^{\mathbf{H}}\mathbf{D}_{T}^{\alpha,\beta;\psi} \left({}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi \right) \right\|_{L^p}.$$

Moreover, if $\alpha > 1/p$, $0 \leq \beta \leq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|\xi\|_{\infty} \leq \frac{(\psi(T) - \psi(0))^{\alpha - 1/p}}{\Gamma(\alpha) \left[(\alpha+1) q + 1 \right]^{1/q}} \frac{[\psi(T)]^{\alpha}}{\Gamma(\alpha+1)} \left\| {}^{\mathbf{H}}\mathbf{D}_{T}^{\alpha,\beta;\psi} \left({}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi \right) \right\|_{L^p}.$$

Theorem 5 (SOUSA; ZUO; O'REGAN, 2021; SOUSA; TAVARES; LEDESMA, 2021; SOUSA et al., 2022) *Let $\alpha \in (0,1)$. Then the embedding $\mathbb{H}_p^{\alpha,\beta;\psi} \hookrightarrow L^p([0,T], \mathbb{R})$ is compact.*

Proposition 6 (SOUSA; ZUO; O'REGAN, 2021; SOUSA; TAVARES; LEDESMA, 2021; SOUSA et al., 2022) *The space $\mathbb{H}_p^{\alpha,\beta;\psi}$ is compactly embedded in $C([0,T], \mathbb{R})$.*

Lemma 7 (SOUSA; ZUO; O'REGAN, 2021; SOUSA; TAVARES; LEDESMA, 2021; SOUSA et al., 2022) *Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$, $u \in \mathbb{H}_p^{\alpha,\beta;\psi}([0,T], \mathbb{R})$. Then $\left[\mathbf{I}_{0+}^{1-\alpha;\psi} u(t) \right]_{t=0} = 0$.*

Proposition 8 (SOUSA; ZUO; O'REGAN, 2021; SOUSA; TAVARES; LEDESMA, 2021; SOUSA et al., 2022) *Consider $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$, $[\psi'(t)]^q \leq \psi'(t)$ for all $t \in [0,T]$ and all $q \geq 1$ with $1 < p \leq \infty$. For all $u \in \mathbb{H}_p^{\alpha,\beta;\psi}$, if $\alpha > 1/p$ it holds that $\mathbf{I}_{0+}^{\alpha;\psi} \left({}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u(t) \right) = u(t)$. Moreover, the inclusion $\mathbb{H}_p^{\alpha,\beta;\psi} \subset C([0,T], \mathbb{R})$ holds.*

Proposition 9 (SOUSA; ZUO; O'REGAN, 2021; SOUSA; TAVARES; LEDESMA, 2021; SOUSA et al., 2022) *Let $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$ and $1 < p < \infty$. Assume that $\alpha > 1/p$ and the sequence $\{\xi_k\}$ converges weakly to ξ in $\mathbb{H}_p^{\alpha,\beta;\psi}$ i.e., $\xi_k \rightharpoonup \xi$. Then $\xi_k \to \xi$ in $C([0,T], \mathbb{R})$, i.e., $\|\xi - \xi_k\|_{\infty} \to 0$ as $k \to \infty$.*

Next we present the Harnack's inequality in the fractional sense with respect to another function.

Theorem 10 (SOUSA; ZUO; O'REGAN, 2021; SOUSA; TAVARES; LEDESMA, 2021; SOUSA et al., 2022) *Let $t_* \geq 0, 0 < \sigma_1 < \sigma_2 < \sigma_3$ and $\rho > 0$. Let further $\alpha \in (0,1)$, $0 \leq \beta \leq 1$, $\psi(0) = 0$ and $\xi_0 \geq 0$. Then for any function $u \in Z(t_*, t_* + \sigma_3 \rho)$ and that satisfies*

$$\partial_t^{\alpha,\beta;\psi} (\xi - \xi_0)(t) = 0, \;\; a.a.\, t \in (t_*, t_* + \sigma_3 \rho) \tag{8}$$

there holds the inequality

$$\sup_{W-} \xi \leq \sigma_3 \sigma_1 \inf_{W+} \xi \tag{9}$$

where $W- = (t_* + \sigma_1 \rho, t_* + \sigma_2 \rho)$ and $W+ = (t_* + \sigma_2 \rho, t_* + \sigma_3 \rho)$.

Proposition 11 (SOUSA; ZUO; O'REGAN, 2021; SOUSA; TAVARES; LEDESMA, 2021; SOUSA et al., 2022) *Let* $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$ *and* $1 < p < \infty$. *The fractional derivative space* $\mathbb{H}_p^{\alpha,\beta;\psi}$ *is a reflexive and separable Banach space.*

Theorem 12 (SOUSA; ZUO; O'REGAN, 2021; SOUSA; TAVARES; LEDESMA, 2021; SOUSA et al., 2022) *The space* $\left(\mathbb{H}_p^{\alpha,\beta,\psi}, \|\cdot\|_{\mathbb{H}_p^{\alpha,\beta,\psi}}\right)$ *is uniformly convex.*

Proof 13 *Indeed, let* $p \in [2,\infty)$. *Then for each* $z, w \in \mathbb{R}$, *it holds*

$$\left|\frac{z+w}{2}\right|^p + \left|\frac{z-w}{2}\right|^p \leqslant \frac{1}{2}(|z|^p + |w|^p).$$

Let $\xi, \zeta \in \mathbb{H}_p^{\alpha,\beta,\psi}$ *satisfy* $\|\xi\|_{\mathbb{H}_p^{\alpha,\beta,\psi}} = \|\zeta\|_{\mathbb{H}_p^{\alpha,\beta,\psi}} = 1$, *and* $\|\xi-\zeta\|_{\mathbb{H}_p^{\alpha,\beta,\psi}} \geqslant \varepsilon \in (0,2]$. *Then, we have*

$$\begin{aligned}
\left\|\frac{\xi+\zeta}{2}\right\|^p_{\mathbb{H}_p^{\alpha,\beta,\psi}} + \left\|\frac{\xi-\zeta}{2}\right\|^p_{\mathbb{H}_p^{\alpha,\beta,\psi}} &= \int_0^T \left(\left|\frac{{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x) +^{\mathbf{H}} \mathbf{D}_{0^+}^{\alpha,\beta,\psi}\zeta(x)^p}{2}\right|\right) dx \\
&+ \int_0^T \left(\left|\frac{{}^{\mathbf{H}}\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x) +^{\mathbf{H}} \mathbf{D}_{0^+}^{\alpha,\beta,\psi}\zeta(x)^p}{2}\right|\right) dx \\
&\leqslant \int_0^T \frac{1}{2}\left(|{}^{\mathbf{H}}\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x)|^p + |{}^{\mathbf{H}}\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\zeta(x)|^p\right) dx \\
&= \frac{1}{2}\left(\|\xi\|_{\mathbb{H}_p^{\alpha,\beta,\psi}} + \|\zeta\|_{\mathbb{H}_p^{\alpha,\beta,\psi}}\right) = 1
\end{aligned}$$

which yields

$$\left\|\frac{\xi+\zeta}{2}\right\|^p_{\mathbb{H}_p^{\alpha,\beta,\psi}} \leqslant 1 - \left(\frac{\varepsilon}{2}\right)^p. \tag{10}$$

On the other hand, if $p \in (1,2)$ *then for each* $z, w \in \mathbb{R}$ *it holds*

$$\left|\frac{z+w}{2}\right|^{p'} + \left|\frac{z-w}{2}\right|^{p'} \leqslant \left(\frac{1}{2}(|z|^p + |w|^p)\right)^{\frac{1}{p-1}}. \tag{11}$$

A straight forward computation proves that if $v \in \mathbb{H}_p^{\alpha,\beta,\psi}$ *then* $\left\||{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\zeta|^p\right\|_{\mathbb{H}_{p-1}^{\alpha,\beta,\psi}} = \|\zeta\|^{p'}_{\mathbb{H}_p^{\alpha,\beta,\psi}}$.

Let $\zeta_1, \zeta_2 \in \mathbb{H}_p^{\alpha,\beta,\psi}$ *then* $\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\zeta_1\right|^{p'}, \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\zeta_2\right|^{p'} \in L^{p-1}([0,T])$ *with* $0 < p-1 < 1$ *and according to*

$$\left\||{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\zeta_1|^{p'} + |{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\zeta_2|^{p'}\right\|_{\mathbb{H}_{p-1}^{\alpha,\beta,\psi}} \geqslant \left\||{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\zeta_1|^{p'}\right\|_{\mathbb{H}_{p-1}^{\alpha,\beta,\psi}} + \left\||{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\zeta_2|^{p'}\right\|_{\mathbb{H}_{p-1}^{\alpha,\beta,\psi}} \tag{12}$$

consequently

$$
\begin{aligned}
&\left\|\frac{\zeta_1+\zeta_2}{2}\right\|^p_{\mathbb{H}_p^{\alpha,\beta,\psi}}+\left\|\frac{\zeta_1-\zeta_2}{2}\right\|^p_{\mathbb{H}_p^{\alpha,\beta,\psi}}\\
=\;& \left\|\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\left(\frac{\zeta_1+\zeta_2}{2}\right)\right|^{p'}\right\|_{\mathbb{H}_{p-1}^{\alpha,\beta,\psi}}+\left\|\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\left(\frac{\zeta_1-\zeta_2}{2}\right)\right|^{p'}\right\|_{\mathbb{H}_{p-1}^{\alpha,\beta,\psi}}\\
\leqslant\;& \left\|\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\left(\frac{\zeta_1+\zeta_2}{2}\right)\right|^{p'}+\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\left(\frac{\zeta_1-\zeta_2}{2}\right)\right|^{p'}\right\|_{\mathbb{H}_{p-1}^{\alpha,\beta,\psi}}\\
=\;& \left[\int_0^T\left(\left|\frac{{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\zeta_1+{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\zeta_2}{2}\right|^{p'}+\left|\frac{{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\zeta_1-{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\zeta_2}{2}\right|^{p'}\right)^{p-1}dx\right]^{\frac{1}{p-1}}\\
\leqslant\;& \left[\frac{1}{2}\int_0^T\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\zeta_1\right|^p+\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\zeta_2\right|^p\right)dx\right]^{\frac{1}{p-1}}\\
=\;& \left(\frac{1}{2}\|\zeta_1\|^p_{\mathbb{H}_p^{\alpha,\beta,\psi}}+\frac{1}{2}\|\zeta_2\|^p_{\mathbb{H}_p^{\alpha,\beta,\psi}}\right)^{\frac{1}{p-1}}.
\end{aligned}\tag{13}
$$

For $\xi,\zeta\in\mathbb{H}_p^{\alpha,\beta,\psi}$ *with* $\|\xi\|_{\mathbb{H}_p^{\alpha,\beta,\psi}}=\|\zeta\|_{\mathbb{H}_p^{\alpha,\beta,\psi}}=1$ *and* $\|\xi-\zeta\|_{\mathbb{H}_p^{\alpha,\beta,\psi}}\geqslant\varepsilon\in(0,2]$, *we have*

$$\left\|\frac{\xi+\zeta}{2}\right\|^{p'}\leqslant 1-\left(\frac{\varepsilon}{2}\right)^{p'}.\tag{14}$$

From Eq.(10) *and* Eq.(14) *in either case there exists* $\delta(\varepsilon)>0$ *such that* $\|\xi+\zeta\|_{\mathbb{H}_p^{\alpha,\beta,\psi}}\leqslant 2(1-\delta(\varepsilon))$.

Remark: The variational structure discussed here were through the weightless ψ-fractional space. The results presented here are for knowledge and allow to discuss the existence, regularity, multiplicity of solutions of fractional differential equations. However, we are already working on some works with the weighted ψ-fractional space.

Below are some relevant points about the results presented above:

1. First, that the above results characterize a variational structure which allows us to investigate results about the existence, multiplicity, regularity of solutions of fractional differential equations with variational problems.

2. Note that the variational structure presented was investigated for the ψ-Hilfer fractional derivative. In this sense, all their respective particular cases also detain from the above results, which implies a particular variational structure for the respective fractional derivative.

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