Revista Eletrônica Paulista de Matemática

ISSN 2316-9664
v. 22, n. 2, set. 2022
Edição Brazilian Symposium on Fractional Calculus

Fractional growth models: Malthus and Verhulst

Junior Cesar Alves Soares
University State of Mato Grosso
juniorcasoares@unemat.br

Stefânia Jarosz
University of Campinas
ra082809@ime.unicamp.br

Felix Silva Costa
State University of Maranhão
felix@cecen.uema.br

Abstract

This article presents fractional versions for Malthus and Verhulst equations, based on the κ-Caputo fractional derivative. Regarding the Malthus model, we present the analytical solution for the κ-fractional equation. For the logistic equation, we present a numerical solution for $\kappa \to 1$ and compare our results with analytical solutions found in literature. Since the numerical results obtained are different from the ones already obtained analytically, we aim to raise some discussion on this matter.

Keywords: Fractional Malthus equation. Fractional logistic equation. Mittag-leffer fuction. κ-Caputo fractional derivative.

1 Introduction

Fractional Calculus (FC) has been used in several areas such as biology, physics, economics, medicine and engineering (TARASOV, 2019). Since its appearance in 1695 several formulations for the fractional operator have been proposed (OLIVEIRA; MACHADO, 2014). More recently, three classes for fractional operators have been proposed, as well as criteria for a fractional operator to be called a fractional derivative (ORTIGUEIRA; MACHADO, 2015; TEODORO; OLIVEIRA; OLIVEIRA, 2017).

One of the most used equations for the study of population growth dynamics and several other applications is the logistic equation, introduced by Verhulst (VERHULST, 1838) in 1938. In his own words,

> "We know that the famous Malthus showed the principle that the human population tends to grow in a geometric progression so as to double after a certain period of time, for example every twenty five years. This proposition is beyond dispute if abstraction is made of the increasing difficulty to find food [. . .]. The virtual increase of the population is therefore limited by the size and the fertility of the country. As a result, the population gets closer and closer to a steady state (VERHULST, 1838)."

Based on this idea, Verhulst formulated a model that has been widely used since then in several applications from biology, physics and economics.

In ref. (EL-SAYED; EL-MESIRY; EL-SAKA, 2007) the authors discuss the stability, existence, uniqueness, and numerical solution of the fractional-order logistic equation. In ref. (WEST, 2015) a general technique to solve fractional logistic equation is proposed and applied to find an analytical solution. However, in ref. (AREA; LOSADA; NIETO, 2016) it is proved that the solution found in the latter is not an exact solution for the fractional logistic equation.

The logistic equation has been used in economic model of natural growth in a competitive environment (TARASOVA; TARASOV, 2017). In (NIETO, 2022) the logistic differential equation of fractional order and non-singular kernel is studied and an analytical solution is obtained.

Many authors have investigated the solution of the logistic equation either from a numerical or analytical point of view, such as (EL-SAYED; BOULAARAS; SWEILAM, 2021; ALCÁNTARA-LÓPEZ et al., 2021; AREA; NIETO, 2021; D'OVIDIO; LORETI; AHRABI, 2018; GUPTA, 2022; JAFARI et al., 2021; KAHARUDDIN; PHANG; JAMAIAN, 2020; KURODA et al., 2017; MAHDY; MOHAMED; MTAWA, 2015; ORTIGUEIRA; BENGOCHEA, 2017).

In this work we present a resolution of the κ-fractional logistics equation by means of the Sumudu transform method, recovering the order equation solution. In addition, we present some important properties of the fractional derivative including the Sumudu transform of the κ- fractional derivative.

2 Preliminaries

To introduce a new fractional operator, this section recovers the concepts of κ-gamma function and κ-beta function and their properties.

Definition 1 *(DIAZ; PARIGUAN, 2007) Let $z \in \mathbb{C}$, $Re(z) > 0$ and $k > 0$, k-gamma function is defined:*

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{-\frac{t^k}{k}} dt. \tag{1}$$

Note that in the limit $k \to 1$ recovers $\Gamma_k(z) = \Gamma(z)$. Some relationships are given in (WANG, 2016):

$$\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right); \tag{2}$$

$$\Gamma_k(k) = 1; \tag{3}$$

$$\Gamma_k(z+k) = z\Gamma_k(z); \tag{4}$$

$$\Gamma_k(z)\Gamma_k(k-z) = \frac{\pi}{k\sin\left(\frac{\pi z}{k}\right)}. \tag{5}$$

Definition 2 *(DIAZ; PARIGUAN, 2007) Let $z \in \mathbb{C}$, $k \in \mathbb{R}$ and $m \in \mathbb{N}^+$, k-Pochhammer symbol is introduced by*

$$(z)_{m,k} = z(z+k)(z+2k)\dots(z+(m-1)k). \tag{6}$$

The expression in Eq.(6) **can be rewritten in terms of k-gamma function, considering the restrictions imposed in Definition 1**:

$$(z)_{m,k} = \frac{\Gamma_k(z+mk)}{\Gamma_k(z)}. \tag{7}$$

Using Eq.(7), the $\Gamma_k(.)$ function is defined by

$$\Gamma_k(z) = \lim_{n\to\infty}\frac{n!k^n(nk)^{\frac{z}{k}-1}}{(z)_{n,k}}. \tag{8}$$

Definition 3 *(DIAZ; PARIGUAN, 2007) Let $z, y \in \mathbb{C}$, $Re(z) > 0$, $Re(y) > 0$, and $k > 0$, k-beta function is defined by:*

$$B_k(y,z) = \frac{1}{k}\int_0^1 u^{\frac{y}{k}-1}(1-u)^{\frac{z}{k}-1}du. \tag{9}$$

A relation between the k-beta function and k-gamma function, similar to the classical ones, can be written as it follows:

$$B_k(y,z) = \frac{\Gamma_k(y)\Gamma_k(z)}{\Gamma_k(y+z)}. \tag{10}$$

Definition 4 *Considering the following set of functions,*

$$A = \left\{f(t)/\exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{|t|/\tau_i}, \text{ if } t \in (-1)^j \times [0,\infty)\right\},$$

the Sumudu Transform is defined as (WATUGALA, 1993)

$$F(u) := \mathcal{S}(f(t)) = \int_0^\infty f(ut)e^{-t}dt,\ u \in (-\tau_1, \tau_2)\,. \tag{11}$$

Lemma 1 *(BELGACEM; KARABALLI; KALLA, 2003) Let $f(t)$ and $g(t)$ be in A and Sumudu transform $F(u)$ and $G(u)$, respectively. Then the Sumudu transform of the convolution of f and g is:*

$$\mathcal{S}((f * g)(t)) = uF(u)G(u)\cdot \tag{12}$$

In (ALKAHTANI; GULATI; KILIÇMAN, 2016) the authors present the following result:

Lemma 2 *In the complex plane $\mathbb{C}$, for any $\Re(\alpha) > 0, \Re(\beta) > 0$ and $\omega \in \mathbb{C}$*

$$\mathcal{S}\left[t^{\alpha-1} E^{\delta}_{\beta,\alpha}\left(\omega t^{\beta}\right)\right] = u^{\alpha-1}\left(1 - \omega u^{\beta}\right)^{-\delta}. \tag{13}$$

3 (k, Ψ)-Riemann-Liouville fractional integrals

In this section, we introduce the (k, Ψ)-fractional integrals by means of a k-fractional integral which generalizes the classical Riemann-Liouville fractional integral.

Definition 5 **((k, Ψ)-Riemann-Liouville fractional integrals)** *(KWUN et al., 2018) Let $f \in L^1[a, b]$, $k > 0$ and the function $\Psi(x)$ increasing and positive monotone on $(a, b]$ with continuous derivative $\Psi'(x)$ on (a, b). The left- and right-sided (k, Ψ)-Riemann-Liouville fractionals integrals of order $\gamma > 0$, of a function f with respect to Ψ on $[a, b]$ are defined by*

$$\left({}_k\mathcal{J}^{\gamma}_{a^+;\Psi} f\right)(x) := \frac{1}{k\Gamma_k(\gamma)} \int_a^x \frac{\Psi'(\xi) f(\xi)}{[\Psi(x) - \Psi(\xi)]^{1-\frac{\gamma}{k}}} \mathrm{d}\xi, \quad x > a,$$

and

$$\left({}_k\mathcal{J}^{\gamma}_{b^-;\Psi} f\right)(x) := \frac{1}{k\Gamma_k(\gamma)} \int_x^b \frac{\Psi'(\xi) f(\xi)}{[\Psi(\xi) - \Psi(x)]^{1-\frac{\gamma}{k}}} \mathrm{d}\xi, \quad x < b,$$

respectively.

Set $\Psi(x) = x^{\rho}/\rho$ with $\rho > 0$ and $x \geq 0$. Substituting $\Psi(x)$ in the expressions given in Definition 5 and rearranging we obtain the left- and right-sided k-fractional integrals

$$\left({}^{\rho}_{k}\mathcal{J}^{\gamma}_{a^+}\varphi\right)(x) = \frac{\rho^{1-\frac{\gamma}{k}}}{k\Gamma_k(\gamma)} \int_a^x \frac{t^{\rho-1}\varphi(t)}{(x^{\rho} - t^{\rho})^{1-\frac{\gamma}{k}}} \mathrm{d}t, \quad x > a, \tag{14}$$

and

$$\left({}^{\rho}_{k}\mathcal{J}^{\gamma}_{b^-}\varphi\right)(x) = \frac{\rho^{1-\frac{\gamma}{k}}}{k\Gamma_k(\gamma)} \int_x^b \frac{t^{\rho-1}\varphi(t)}{(t^{\rho} - x^{\rho})^{1-\frac{\gamma}{k}}} \mathrm{d}t, \quad x < b, \tag{15}$$

respectively.

Definition 6 (**(κ, ρ)-Hilfer fractional derivative**) *(OLIVEIRA, 2018) Let γ denote the order of the fractional derivative, $0 < \gamma \leq 1$, and let β denote its type, with $0 \leq \beta \leq 1$. The (left- and right-sided) fractional derivatives of φ with respect to x, for $\rho > 0$, are given by*

$$\left({}^{\rho}_{k}D^{\gamma,\beta}_{a\pm}\varphi\right)(x) = \left(\pm{}^{\rho}_{k}\mathcal{J}^{\beta(k-\gamma)}_{a\pm}\left(x^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}x}\right)\left(k\,{}^{\rho}_{k}\mathcal{J}^{(1-\beta)(k-\gamma)}_{a\pm}\varphi\right)\right)(x), \tag{16}$$

for functions such that the expression on the right hand side exists.

The k-fractional derivative generalizes several classical fractional operators (OLIVEIRA, 2018), specifically, when $(k, \rho) \to (1, 1)$ we obtain Riemann-Liouville and Caputo operators (associated with translations) and Hadamard operator (associated with dilations) as particular cases.

4 κ-Hilfer derivative and particular cases

The κ-Hilfer fractional derivative is introduced considering $\rho \to 1$ in Eq. (16):

$$\left({}_{k}D^{\gamma,\beta}_{a\pm}\varphi\right)(x) = \left(\pm{}_{k}\mathcal{J}^{\beta(k-\gamma)}_{a\pm}\left(\kappa\frac{\mathrm{d}}{\mathrm{d}x}\right)\left({}_{k}\mathcal{J}^{(1-\beta)(k-\gamma)}_{a\pm}\varphi\right)\right)(x). \tag{17}$$

- For $\beta \to 0$ the κ-Hilfer fractional derivative reduces the κ-Riemann-Liouville fractional derivative given by:

$$\left({}^{RL}_{k}D^{\gamma}_{a\pm}\varphi\right)(x) = \left(\kappa\frac{\mathrm{d}}{\mathrm{d}x}\right)\left(\pm{}_{k}\mathcal{J}^{(k-\gamma)}_{a\pm}\varphi\right)(x). \tag{18}$$

  Using Definition (5) and $a \to 0$ we can be rewrite Eq. (18) as

$$\left({}^{RL}_{k}D^{\gamma}\varphi\right)(x) = \frac{1}{\Gamma_k(k-\gamma)}\frac{d}{dx}\int_0^x (x-t)^{\frac{k-\gamma}{k}-1}\,\varphi(t)\,\mathrm{d}t. \tag{19}$$

- For $\beta \to 1$ the κ-Hilfer fractional derivative reduces to κ-Caputo fractional derivative described by:

$$\left({}^{C}_{k}D^{\gamma}_{a\pm}\varphi\right)(x) = \left(\pm{}_{k}\mathcal{J}^{(k-\gamma)}_{a\pm}k\varphi'\right)(x). \tag{20}$$

  Using Definition (5) and $a \to 0$ we can be rewrite (20) as

$$\left({}^{C}_{k}D^{\gamma}\varphi\right)(x) = \frac{1}{\Gamma_k(k-\gamma)}\int_0^x (x-t)^{\frac{k-\gamma}{k}-1}\,\varphi'(t)\,\mathrm{d}t\cdot \tag{21}$$

The particular case $\kappa \to 1$ recovers the Caputo fractional derivative:

$$\left({}^{C}D^{\gamma}\varphi\right)(x) = \frac{1}{\Gamma(1-\gamma)}\int_0^x (x-t)^{-\gamma}\,\varphi'(t)\,\mathrm{d}t. \tag{22}$$

Proposition 1 *The Sumudu transform of the κ-Hilfer fractional derivative with* $0 < \gamma < 1$ *and* $0 < \beta < 1$ *is given by*

$$\mathcal{S}\left[\left({}_{k}D_{0}^{\gamma,\beta}\varphi\right)(x)\right] = (uk^{-1})^{-\frac{\gamma}{\kappa}}\mathcal{S}\left[\varphi(x)\right] - (uk^{-1})^{\beta(\frac{\kappa-\gamma}{\kappa})-1}\left({}_{\kappa}\mathcal{J}^{(1-\beta)(k-\gamma)}\varphi(x)\right)\Big|_{x=0^{+}}. \tag{23}$$

Proof 1

$$\begin{aligned}\mathcal{S}\left[\left({}_{k}D_{0}^{\gamma,\beta}\varphi\right)(x)\right] &= \mathcal{S}\left[\left({}_{k}\mathcal{J}^{\beta(k-\gamma)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)(k\ {}_{k}\mathcal{J}^{(1-\beta)(k-\gamma)}\varphi)\right)(x)\right]\\
&= \mathcal{S}\left[\int_{0}^{x}\frac{(x-t)^{\frac{\beta(\kappa-\gamma)}{\kappa}-1}}{\kappa\Gamma_{\kappa}(\beta(\kappa-\gamma))}\left(\frac{\mathrm{d}}{\mathrm{d}x}(\kappa\ {}_{\kappa}\mathcal{J}^{(1-\beta)(k-\gamma)}\varphi)\right)(t)\ dt\right]\\
&= \mathcal{S}\left[\frac{x^{\frac{\beta(\kappa-\gamma)}{\kappa}-1}}{\kappa\Gamma_{\kappa}(\beta(\kappa-\gamma))} * \left(\frac{\mathrm{d}}{\mathrm{d}x}(k\ {}_{k}\mathcal{J}^{(1-\beta)(k-\gamma)}\varphi)\right)(t)\right]\\
&= u\ \mathcal{S}\left[\frac{x^{\frac{\beta(\kappa-\gamma)}{\kappa}-1}}{\kappa\Gamma_{\kappa}(\beta(\kappa-\gamma))}\right]\mathcal{S}\left[\left(\frac{\mathrm{d}}{\mathrm{d}x}(k\ {}_{\kappa}\mathcal{J}^{(1-\beta)(k-\gamma)}\varphi)\right)(t)\right].\end{aligned}$$

From that, we can express the Sumudu transforms of each expression above as

$$\begin{aligned}\mathcal{S}\left[\frac{x^{\frac{\beta(\kappa-\gamma)}{\kappa}-1}}{\kappa\Gamma_{\kappa}(\beta(\kappa-\gamma))}\right] &= \int_{0}^{\infty}\frac{(ux)^{\frac{\beta(\kappa-\gamma)}{\kappa}-1}}{\kappa\Gamma_{\kappa}(\beta(\kappa-\gamma))}e^{-t}\ dt = \frac{u^{\frac{\beta(\kappa-\gamma)}{\kappa}-1}}{\kappa\Gamma_{k}(\beta(\kappa-\gamma))}\int_{0}^{\infty}x^{\frac{\beta(\kappa-\gamma)}{\kappa}-1}e^{-t}\ dt\\
&= \frac{u^{\frac{\beta(\kappa-\gamma)}{\kappa}-1}}{\kappa\Gamma_{\kappa}(\beta(\kappa-\gamma))}\Gamma\left(\frac{\beta(\kappa-\gamma)}{\kappa}\right).\end{aligned} \tag{24}$$

Using (2) in (24) we obtain

$$\mathcal{S}\left[\frac{x^{\frac{\beta(\kappa-\gamma)}{\kappa}-1}}{\kappa\Gamma_{\kappa}(\beta(\kappa-\gamma))}\right] = \frac{u^{\frac{\beta(\kappa-\gamma)}{\kappa}-1}}{\kappa\Gamma_{\kappa}(\beta(\kappa-\gamma))}\cdot\kappa^{1-\frac{\beta(\kappa-\gamma)}{\kappa}}\cdot\Gamma_{\kappa}\left(\beta(\kappa-\gamma)\right).$$

Thus,

$$\mathcal{S}\left[\frac{x^{\frac{\beta(\kappa-\gamma)}{\kappa}-1}}{\kappa\Gamma_{\kappa}(\beta(\kappa-\gamma))}\right] = \frac{u^{\frac{\beta(\kappa-\gamma)}{\kappa}-1}\cdot\kappa^{1-\frac{\beta(\kappa-\gamma)}{\kappa}}}{\kappa}. \tag{25}$$

Now, we calculate

$$\begin{aligned}\mathcal{S}\left[\left(\frac{\mathrm{d}}{\mathrm{d}x}(k\ {}_{\kappa}\mathcal{J}^{(1-\beta)(k-\gamma)}\varphi)\right)(t)\right] &= \kappa u^{-1}\left(\mathcal{S}\left[\left({}_{\kappa}\mathcal{J}^{(1-\beta)(k-\gamma)}\varphi\right)(t)\right] - \left({}_{\kappa}\mathcal{J}^{(1-\beta)(k-\gamma)}\varphi(t)\right)\Big|_{t=0^{+}}\right)\\
&= \kappa u^{-1}\left(\mathcal{S}\left[\int_{0}^{x}\frac{(x-t)^{\frac{(1-\beta)(\kappa-\gamma)}{\kappa}-1}}{\kappa\Gamma_{\kappa}((1-\beta)(\kappa-\gamma))}\varphi(t)\ dt\right] - \left({}_{\kappa}\mathcal{J}^{(1-\beta)(k-\gamma)}\varphi(t)\right)\Big|_{t=0^{+}}\right)\\
&= \kappa u^{-1}\left(\mathcal{S}\left[\frac{x^{\frac{(1-\beta)(\kappa-\gamma)}{\kappa}-1}}{\kappa\Gamma_{\kappa}((1-\beta)(\kappa-\gamma))} * \varphi(x)\right] - \left({}_{\kappa}\mathcal{J}^{(1-\beta)(k-\gamma)}\varphi(t)\right)\Big|_{t=0^{+}}\right)\\
&= \kappa u^{-1}\left(u\mathcal{S}\left[\frac{x^{\frac{(1-\beta)(\kappa-\gamma)}{\kappa}-1}}{\kappa\Gamma_{\kappa}((1-\beta)(\kappa-\gamma))}\right]\mathcal{S}\left[\varphi(x)\right] - \left({}_{\kappa}\mathcal{J}^{(1-\beta)(k-\gamma)}\varphi(t)\right)\Big|_{t=0^{+}}\right).\end{aligned}$$

Therefore,

$$\mathcal{S}\left[\left(\frac{\mathrm{d}}{\mathrm{d}x}(k\ {}_{\kappa}\mathcal{J}^{(1-\beta)(k-\gamma)}\varphi)\right)(t)\right] = \left[(u\cdot\kappa^{-1})^{-\frac{\gamma}{\kappa}}\mathcal{S}\left[\varphi(x)\right] - \kappa u^{-1}\left({}_{\kappa}\mathcal{J}^{(1-\beta)(k-\gamma)}\varphi(t)\right)\Big|_{t=0^{+}}\right]\cdot \quad (26)$$

Finally, it follows from Eqs. (25) and (26) that

$$\mathcal{S}\left[\left({}_{k}D_{0}^{\gamma,\beta}\varphi\right)(x)\right] = (u\cdot k^{-1})^{-\frac{\gamma}{\kappa}}\cdot\mathcal{S}\left[\varphi(x)\right] - (u\cdot k^{-1})^{\beta(\frac{\kappa-\gamma}{\kappa})-1}\left({}_{\kappa}\mathcal{J}^{(1-\beta)(k-\gamma)}\varphi(x)\right)\Big|_{x=0^{+}}\cdot \quad (27)$$

The special fractional operator given by Eq. (17) is a interpolation between κ-Riemann-Liouville fractional derivative and κ-Caputo fractional derivative in agree as properties to follow:

Corollary 1 *The Sumudu transform of the k-Caputo fractional derivative with $0<\gamma<1$ is given by*

$$\mathcal{S}\left[\left({}_{k}^{C}D^{\gamma}\varphi\right)(x)\right] = (uk^{-1})^{-\frac{\gamma}{k}}\left[F(u)-f(0)\right]\cdot \quad (28)$$

5 The exponential and logistic growth

Malthus equation

The Malthusian growth model is defined by

$$\frac{d}{dt}P(t) = rP(t), \quad (29)$$

where r is growth rate. The analytical solution to equation Eq.(29), with initial condition $P(0)=P_0$ is given by

$$P(t) = P_0\mathrm{e}^{rt}. \quad (30)$$

Logistic equation

The classical logistic equation is given by

$$\frac{dP(t)}{dt} = r\left(1-\frac{P(t)}{\lambda}\right)P(t), \quad (31)$$

where $r>0$ and λ are Malthusian parameter related with the maximum growth rate and carrying capacity, respectively. $P(t)$ is the size of population growth at a time t.

Taking the change of variables $\varphi(t)=\dfrac{1}{P(t)}$, the logistic equation (31) can be rewritten as

$$\frac{d\varphi(t)}{dt} = \frac{r}{\lambda}\left(1-\lambda\varphi(t)\right),\ t\geq 0, \quad (32)$$

with solution given by

$$P(t) = \frac{\lambda P_0}{P_0+(\lambda-P_0)\mathrm{e}^{-rt}}\cdot \quad (33)$$

6 Fractional models: Malthus and Logistic

6.1 Fractional Malthus equation

R. Almeida et al. (ALMEIDA; BASTOS; MONTEIRO, 2016) proposed a fractional Malthus equation based on Caputo fractional derivative, as it follows:

$$\left({}^{C}\mathcal{D}^{\alpha}P\right)(t) = rP(t), \tag{34}$$

with $0 < \alpha < 1$. The analytical solution considering the initial condition $P_0 = P(0)$ is

$$P(t) = P_0 \mathrm{E}_{\alpha}(rt^{\alpha}), \tag{35}$$

where $\mathrm{E}_{\alpha}(\cdot)$ is the Mittag-Leffler function one parameter α.

6.2 Fractional logistic equation

In Refs. (CAMARGO; BRUNO-ALFONSO, 2012; TEODORO, 2019; THEODORO; CAMARGO, 2020; VARALTA; GOMES; CAMARGO, 2014) , a fractional version using Caputo derivative is presented:

$$({}^{C}\mathcal{D}^{\alpha}P)(t) = r\left(1 - \frac{P(t)}{\lambda}\right), \tag{36}$$

where $0 < \alpha \leq 1$, $t > 0$ and $P(t)$ is a continuous function, with solution given by

$$P(t) = \frac{\lambda P_0}{P_0 + (\lambda - P_0)E_{\alpha}(-rt^{\alpha})}. \tag{37}$$

The solution given by Eq. (37) is found using Laplace transform and its properties with respect Mittag-Leffler function.

It is important to emphasize that

$$\lim_{t\to\infty} \frac{\lambda P_0}{P_0 + (\lambda - P_0)E_{\alpha}(-rt^{\alpha})} = \lambda.$$

Furthermore, when $\alpha \to 1$, Eq. (37) recovers the solution of the classical logistic equation, given by Eq. (31).

However, a further analysis in the solution given by (37) shows that a possible analytical solution for the fractional version does not lie in this presented form. This will be discussed in Sec. 9.

7 κ-Fractional Malthus and Logistic Equations

Definition 7 *The Malthus equation based on κ-Caputo fractional derivative is defined by*

$$\left({}^{C}_{\kappa}\mathcal{D}^{\alpha}P\right)(t) = rP(t), \tag{38}$$

with $0 < \alpha < 1$ and $\kappa \geq 1$. Taking into account the initial condition $P_0 = P(0)$, its analytical solution is given by

$$P(t) = P_0 \mathrm{E}_{\frac{\alpha}{\kappa}}(rt^{\frac{\alpha}{\kappa}}). \tag{39}$$

where $\mathrm{E}_{\gamma}(\cdot)$ is the Mittag-Leffler function with one parameter γ.

Definition 8 *The Logistic equation in terms of the κ-Caputo Derivative Operator is given by*

$$\left({}_{k}^{C}\mathcal{D}^{\gamma}P\right)(t) = rP(t)\left(1 - \frac{P(t)}{\lambda}\right). \tag{40}$$

The next step is the linearization of the equation (40) in terms of the κ-Caputo Derivative. In general, the same idea made in the case of integer order is used to find a possible analytical solution of the fractional logistic equation. However, we will see that the solutions are not equivalent and, therefore, solving the logistic equation in the fractional case requires more care in the considerations.

Proposition 2 *The linear equation in terms of κ-Caputo Derivative is given by*

$${}_{k}^{C}D^{\gamma}\varphi(t) = \frac{r}{\lambda}\left(1 - \lambda\varphi(t)\right), \tag{41}$$

where, $0 < \gamma < 1$. Its solution is given by

$$\varphi(t) = \frac{1}{\lambda} + \left(\varphi(0^{+}) - \frac{1}{\lambda}\right)E_{\frac{\gamma}{\kappa}}\left(-r\left(\frac{t}{\kappa}\right)^{\frac{\gamma}{\kappa}}\right). \tag{42}$$

Proof 2 *Applying Sumudu Transform on both sides, we have:*

$$\begin{aligned}
\mathcal{S}\left[{}_{k}D^{\gamma}\varphi(t)\right] &= \mathcal{S}\left[\frac{r}{\lambda}\left(1 - \lambda\varphi(t)\right)\right] \\
\left(uk^{-1}\right)^{-\frac{\gamma}{\kappa}}\left[\Phi(u) - \varphi(0^{+})\right] &= \frac{r}{\lambda} - r\Phi(u) \\
\left(uk^{-1}\right)^{-\frac{\gamma}{\kappa}}\Phi(u) - \left(uk^{-1}\right)^{-\frac{\gamma}{\kappa}}\varphi(0^{+}) &= \frac{r}{\lambda} - r\Phi(u) \\
\Phi(u) - \varphi(0^{+}) &= \left(uk^{-1}\right)^{\frac{\gamma}{\kappa}}\frac{r}{\lambda} - r\left(uk^{-1}\right)^{\frac{\gamma}{\kappa}}\Phi(u).
\end{aligned} \tag{43}$$

Then,

$$\Phi(u) = \frac{\left(uk^{-1}\right)^{\frac{\gamma}{\kappa}}}{1 + r\left(uk^{-1}\right)^{\frac{\gamma}{\kappa}}}\cdot\frac{r}{\lambda} + \frac{\left(uk^{-1}\right)^{\frac{\gamma}{\kappa}}}{r + \left(uk^{-1}\right)^{-\frac{\gamma}{\kappa}}}\cdot\varphi(0^{+}). \tag{44}$$

Applying Sumudu transform via Lemma 2, we have

$$\varphi(t) = \varphi(0^{+})E_{\frac{\gamma}{\kappa}}\left(-r\left(\frac{t}{\kappa}\right)^{\frac{\gamma}{\kappa}}\right) + \frac{r}{\lambda}\left(\frac{t}{\kappa}\right)^{\frac{\gamma}{\gamma}}E_{\frac{\gamma}{\kappa},\frac{\gamma}{\kappa}+1}\left(-r\left(\frac{t}{\kappa}\right)^{\frac{\gamma}{\kappa}}\right).$$

Using the following relation,

$$r\left(\frac{t}{\kappa}\right)^{\frac{\gamma}{\kappa}}E_{\frac{\gamma}{\kappa},\frac{\gamma}{\kappa}+1}\left(-r\left(\frac{t}{\kappa}\right)^{\frac{\gamma}{\kappa}}\right) = 1 - E_{\frac{\gamma}{\kappa}}\left(-r\left(\frac{t}{\kappa}\right)^{\frac{\gamma}{\kappa}}\right), \tag{45}$$

we obtain

$$\varphi(t) = \frac{1}{\lambda} + \left(\varphi(0^{+}) - \frac{1}{\lambda}\right)E_{\frac{\gamma}{\kappa}}\left(-r\left(\frac{t}{\kappa}\right)^{-\frac{\gamma}{\kappa}}\right). \tag{46}$$

It is convenient to write the solution (46) in the original variable, that is, $y(t)$ in this way we obtain

$$y(t) = \frac{\lambda y_0}{y_0 + (\lambda - y_0)E_{\frac{\gamma}{\kappa}}\left(-r\left(\frac{t}{\kappa}\right)^{-\frac{\gamma}{\kappa}}\right)}, \tag{47}$$

where $y_0 = \dfrac{1}{\varphi(0^+)}$ and $\varphi(0^+) > 0$.

Remark-1: In (WEST, 2015), an exact solution for logistic equation is obtained using the Carleman embedding technique. The solution presented was

$$P(t) = \sum_{n=0}^{\infty}\left(\frac{P_0 - 1}{P_0}\right)^n E_\alpha(-nk^\alpha t^\alpha.). \tag{48}$$

In order to validate this exact solution, the author compared it with the numerical solution. However, in ref. (AREA; LOSADA; NIETO, 2016), it is proven that the solution given by Eq. (48) is not in fact a solution to the nonlinear equation given by Eq. (40). We can therefore conclude that the Mittag-Leffer function does not preserve a property arising from the classical exponential function.

This consequence is caused by the violation the semigroup property by the Mittag–Leffler function, i. e., we have $E_\alpha\left[\lambda(t+s)^\alpha\right] \neq E_\alpha\left[\lambda(t)^\alpha\right] E_\alpha\left[\lambda(s)^\alpha\right]$ for $\alpha \in (0,1)$, and real constant λ.

In addition, we can also state that the exact solution given by Eq. (48) only represents a trivial case of the logistic equation.

In the book (TARASOV, 2019) , the authors carry out a detailed study to clarify the difference between the solutions of the following differential equations:

$$\left({}^{C}\mathcal{D}^\alpha P\right)(t) = \nu(\alpha)P(t)(1 - P(t)), \tag{49}$$

and

$$\left({}^{C}\mathcal{D}^\alpha u\right)(t) = \nu(\alpha)(1 - u(t)). \tag{50}$$

The authors claim that Eq.(49) and Eq.(50) have non-equivalents solutions and generalizations of standard out-of-memory models violate the equivalence of fractional differential equations. The authors also argue that a generalization of the logistic equation which uses the entire case solution procedure causes the violation of the chain rule, product rule and other rules necessary for the validity of the solution for the fractional case.

Taking $\kappa = 1$ in Eq.(37), the results which found in references (CAMARGO; BRUNO-ALFONSO, 2012; TEODORO, 2019; THEODORO; CAMARGO, 2020) and (VARALTA; GOMES; CAMARGO, 2014) are recovered.

It is noteworthy that the solution given by Eq. (42) was obtained using the Sumudu transform method, which is both Laplace and Sehu transforms type. In Ref. (COSTA; SOARES; JAROSZ, 2022), a more general Laplace type transform, called *Jafari transform*, is discussed .

8 Discrete models

Here, we present the discretization to Malthus and logistic equations based on Caputo fractional derivative proposed in in Ref. (LI; ZENG, 2015):

$$\mathcal{I}^{\alpha} y_n \approx \widehat{\mathcal{I}}^{\alpha} y_n, \tag{51}$$

$$^{C}\mathcal{D}^{\alpha} y_n \approx {}^{C}\widehat{\mathcal{D}}^{\alpha} y_n, \tag{52}$$

in which $y_n := y(t_n)$, $t_n = nh$, $h > 0$ and the discretizations $\widehat{\mathcal{I}}^{\alpha}$ and $\widehat{\mathcal{D}}^{\alpha}$ are defined by

$$\widehat{\mathcal{I}}^{\alpha} = \frac{h^{\alpha}}{\Gamma(1+\alpha)} \sum_{i=0}^{n-1} b_{n-i}(\alpha) y_{i+1}, \tag{53}$$

$$\widehat{\mathcal{D}}^{\alpha} = \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{i=0}^{n-1} b_{n-i}(1-\alpha)(y_{i+1} - y_i), \tag{54}$$

with weights

$$b_j(\beta) = (j+1)^{\beta} - j^{\beta}. \tag{55}$$

The fractional Discrete models are given by:

- Discrete fractional Malthus:

$$^{C}\widehat{D}^{\gamma}\, P_n = r P_{n-1}; \tag{56}$$

- Discrete fractional logistic:

$$^{C}\widehat{D}^{\gamma} P_n = r P_{n-1}\left(1 - \frac{P_{n-1}}{\lambda}\right). \tag{57}$$

Comparison between Numerical and analytical solutions to fractional models using real data

In Ref. (ALMEIDA; BASTOS; MONTEIRO, 2016), the author used several databases with the world population throughout the centuries, provided by the United Nations, from year 1910 until 2010.Taking the initial value as $P_0 = 1750$, it was obtained the value $r = 1.3501 \times 10^{-2}$ for the classical Malthus model and $r = 3.4399 \times 10^{-3}$ for the fractional Malthus model with $\alpha = 1.393298754843208$.

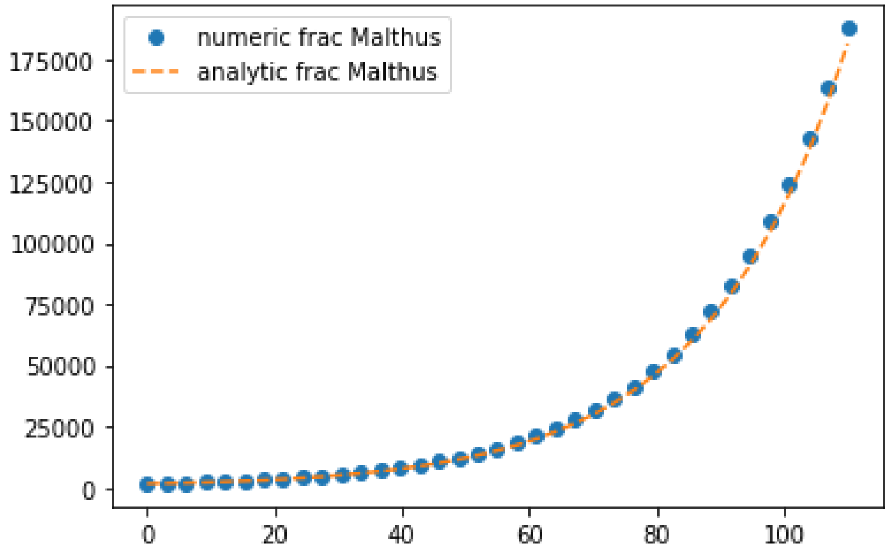

Figure 1: Fractional Malthus models with $\alpha = 1.39$, according to Eqs. (39) and (56).

Fig. 1 illustrates the efficiency of the numerical solution in relation to analytical solution to fractional case.

We will now present another example of the population growth, considering fractional logistic equation. Proposed by D. S. Rodrigues and E. B. Hauser, it is considered a population growth with support capacity (RODRIGUES; HAUSER, 2014). The values used are $r = 0.026$ and $\lambda = 12$ billion.

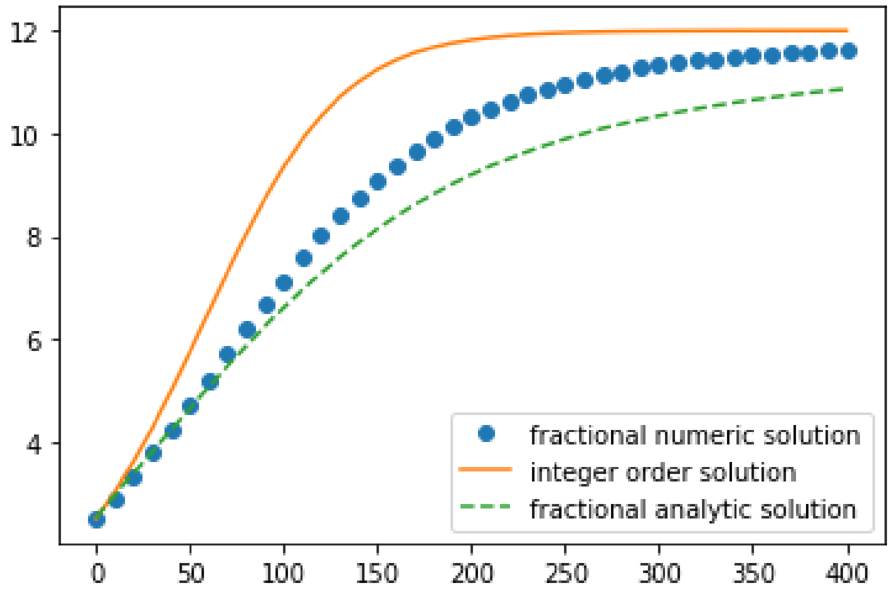

Figure 2: Fractional logistic models with $\alpha = 0.9$, according to Eqs. (37) and (57).

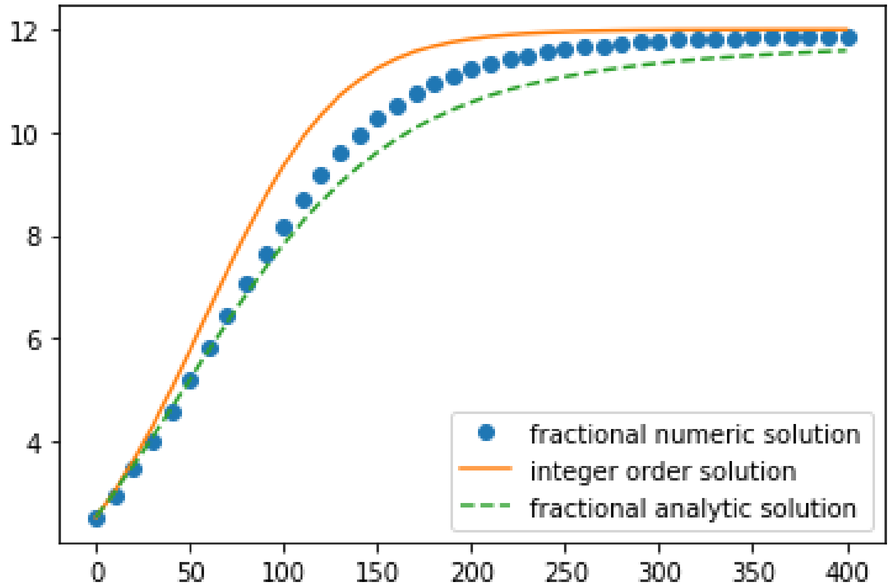

Figure 3: Fractional logistic models with $\alpha = 0.95$, according to Eqs. (37) and (57).

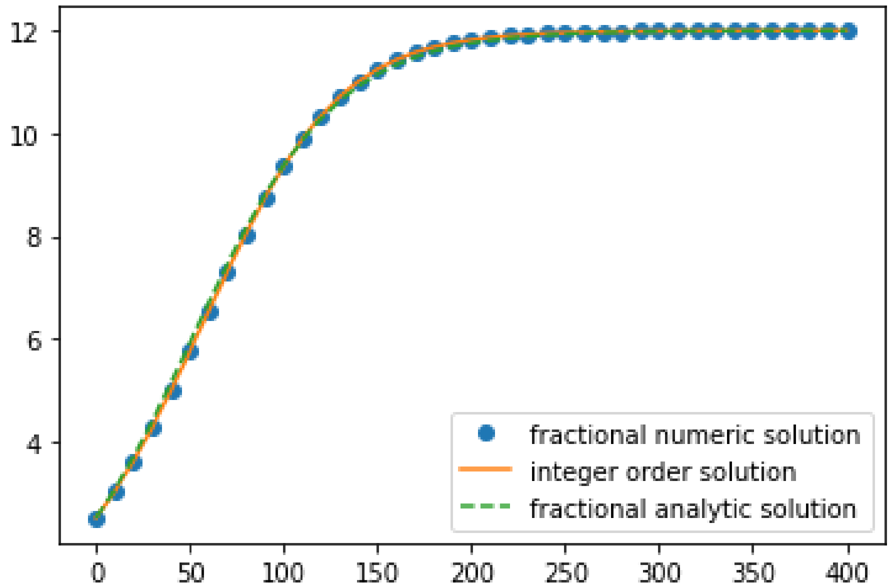

Figure 4: Fractional logistic models with $\alpha = 1$, according to Eqs. (37) and (57).

Figs. 2 and 3 show that Eq. (37) differs from the numerical solution. That leads us to the following questions:

- How efficient is the method used?
- Is there a possibility that the analytical solution given in Eq. (37) is not the solution for the fractional logistic equation ?

Regarding the first question, the method used is based on the discretization of the Caputo fractional derivative via rectangle method, which has been widely applied in several problems involving fractional differential equations, with good results. For the readers interested in the study of the convergence order of the method, more details can be found in (LI; ZENG, 2015).

In the case of the logistic equation, the solution given by Eq. (33) coincides with both the numerical and the proposed analytical given by Eq. (37) for $\alpha = 1$. However, a visual analysis in Figs. 2 and 3 draws our attention to a critical analysis on the proposed analytical solution. In Refs. (AREA; LOSADA; NIETO, 2016; TARASOVA; TARASOV, 2017; NIETO, 2022), some problems in its use are pointed out.

9 Concluding Remarks

In this work, we propose an even more generalized for fractional models by means of the fractional κ-Caputo derivative, with the aim of obtaining special classes of solutions in order to better understand what to expect concerning the behavior of the models. In particular, for the fractional logistic model, the solution proposed in Refs. (CAMARGO; BRUNO-ALFONSO, 2012; TEODORO, 2019; THEODORO; CAMARGO, 2020; VARALTA; GOMES; CAMARGO, 2014) does not seem to be completely suitable for real life applications. This raises an important question: what is the analytical solution for the fractional logistic model? V. E. Tarasov and J. J. Nieto have been proposing some alternatives in search of this answer. We believe that this present work may help to give some direction by comparing two or more kinds of solution and improve both methodologies.

References

ALCÁNTARA-LÓPEZ, F. et al. Fractional growth model applied to covid-19 data. **Mathematics**, v. 9, n. 16, p. 1915, 2021.

ALKAHTANI, B.; GULATI, V.; KILIÇMAN, A. Application of sumudu transform in generalized fractional reaction–diffusion equation. **International Journal of Applied and Computational Mathematics**, São Paulo, v. 2, n. 3, p. 387–394, 2016.

ALMEIDA, R.; BASTOS, N. R. O.; MONTEIRO, M. T. T. Modeling some real phenomena by fractional differential equations. **Mathematical Methods in the Applied Sciences**, São Paulo, v. 39, n. 16, p. 4846–4855, 2016.

AREA, I.; LOSADA, J.; NIETO, J. J. A note on the fractional logistic equation. **Physica A:** statistical mechanics and its applications, Amsterdam, v. 444, p. 182–187, 2016.

AREA, I.; NIETO, J. Power series solution of the fractional logistic equation. **Physica A:** statistical mechanics and its applications, Amsterdam, v. 573, p. 125947, 2021.

BELGACEM, F. B. M.; KARABALLI, A. A.; KALLA, S. L. Analytical investigations of the sumudu transform and applications to integral production equations. **Mathematical problems in Engineering**, London, v. 2003, n. 3, p. 103–118, 2003.

CAMARGO, R. F.; BRUNO-ALFONSO, A. Equacao logistica fracionária. *In*: CONGRESSO DE MATEMATICA APLICADA E COMPUTACIONAL (CMAC NORDESTE), 2012, Natal. **Anais do** [...]. Natal: Sbmac, 2012.

COSTA, F.; SOARES, J. C. A.; JAROSZ, S. Integral transforms of the κ-hilfer fractional derivative. **Authorea**, Hoboken, v. 2, 2022.

DIAZ, R.; PARIGUAN, E. On hypergeometric functions and pochhammer k-symbol. **Divulgaciones MatemáTicas**, v. 15, n. 2, p. 179–192, 2007.

D'OVIDIO, M.; LORETI, P.; AHRABI, S. S. Modified fractional logistic equation. **Physica A:** statistical mechanics and its applications, Amsterdam, v. 505, p. 818–824, 2018.

EL-SAYED, A.; EL-MESIRY, A.; EL-SAKA, H. On the fractional-order logistic equation. **Applied Mathematics Letters**, Amsterdam, v. 20, n. 7, p. 817–823, 2007.

EL-SAYED, A. A. E.; BOULAARAS, S.; SWEILAM, N. Numerical solution of the fractional-order logistic equation via the first-kind dickson polynomials and spectral tau method. **Mathematical Methods in the Applied Sciences**, Wiley Online Library, 2021.

GUPTA, A. A new approach to solve fractional logistic growth model and its numerical simulation. *In*: INTERNACIONAL CONFERENCE ON COMPUTACIONAL INTELLIGENCE AND COMPUTING, jul. 2021, SINGAPORE. Singapore: Springer, 2022. p. 119–129.

JAFARI, H. et al. A numerical study of fractional order population dynamics model. **Results in Physics**, Amsterdam, v. 27, p. 104456, 2021.

KAHARUDDIN, L. N.; PHANG, C.; JAMAIAN, S. S. Solution to the fractional logistic equation by modified eulerian numbers. **The European Physical Journal Plus**, New York, v. 135, n. 2, p. 1–11, 2020.

KURODA, L. K. B. et al. Unexpected behavior of caputo fractional derivative. **Computational and Applied Mathematics**, Rio de Janeiro, v. 36, n. 3, p. 1173–1183, 2017.

KWUN, Y. C. et al. Generalized riemann-liouville k-fractional integrals associated with ostrowski type inequalities and error bounds of hadamard inequalities. **IEEE access**, California, v. 6, p. 64946–64953, 2018.

LI, C.; ZENG, F. **Numerical methods for fractional calculus**. Boca Raton: CRC Press, 2015. v. 24.

MAHDY, A.; MOHAMED, A.; MTAWA, A. Sumudu decomposition method for solving fractional-order logistic differential equation. **Journal of Advances in Mathematics**, London, v. 10, n. 7, 2015.

NIETO, J. J. Solution of a fractional logistic ordinary differential equation. **Applied Mathematics Letters**, Amsterdam, v. 123, p. 107568, 2022.

OLIVEIRA, D. S. **Fractional derivatives**: generalizations. 2018. 105 f. Tese (Doutorado em Matemática Aplicada) — Universidade Estadual de Campinas, Campinas, 2018.

OLIVEIRA, E. C.; MACHADO, J. A. T. A review of definitions for fractional derivatives and integral. **Mathematical Problems in Engineering**, London, v. 2014, 2014.

ORTIGUEIRA, M.; BENGOCHEA, G. A new look at the fractionalization of the logistic equation. **Physica A:** statistical mechanics and its applications, Amsterdam, v. 467, p. 554–561, 2017.

ORTIGUEIRA, M. D.; MACHADO, J. A. T. What is a fractional derivative? **Journal of computational Physics**, Amsterdam, v. 293, p. 4–13, 2015.

RODRIGUES, D. S.; HAUSER, E. B. Modelo logístico de verhulst e métodos numéricos na análise do censo populacional mundial. **Proceeding Series of the Brazilian Society of Computational and Applied Mathematics**, Rio de Janeiro, v. 2, n. 1, 2014.

TARASOV, V. E. **Handbook of fractional calculus with applications**. Berlin: De Gruyter, 2019. v. 5.

TARASOVA, V. V.; TARASOV, V. E. Logistic map with memory from economic model. **Chaos, Solitons & Fractals**, Amsterdam, v. 95, p. 84–91, 2017.

TEODORO, G.; OLIVEIRA, D. S.; OLIVEIRA, E. C. Sobre derivadas fracionárias. **Revista Brasileira de Ensino de Física**, São Paulo, v. 40, 2017.

TEODORO, G. S. **Derivadas Fracionárias**: tipos e critérios de validade. 2019. 181 f. Tese (Doutorado em Matemática Aplicada) — Universidade Estadual de Campinas, Campinas, 2019.

THEODORO, M. M.; CAMARGO, R. F. A study about the solutions of fractional logistic equation. **C. Q. D. – Revista Eletrônica Paulista de Matemática**, Bauru, v. 17, p. 61–70, 2020.

VARALTA, N.; GOMES, A. V.; CAMARGO, R. F. A prelude to the fractional calculus applied to tumor dynamic. **TEMA**, São Carlos, v. 15, p. 211–221, 2014.

VERHULST, P.-F. Notice sur la loi que la population suit dans son accroissement. **Correspondence Mathematique et Physique**, Ghent, v. 10, p. 113–126, 1838.

WANG, W.-S. Some properties of k-gamma and k-beta functions. **ITM Web of Conferences**, Les Ulis, v. 7, p. 07003, 2016.

WATUGALA, G. Sumudu transform: a new integral transform to solve differential equations and control engineering problems. **Integrated Education**, United Kingdom, v. 24, n. 1, p. 35–43, 1993.

WEST, B. J. Exact solution to fractional logistic equation. **Physica A:** statistical mechanics and its applications, Amsterdam, v. 429, p. 103–108, 2015.