Revista Eletrônica
Paulista de Matemática

ISSN 2316-9664
v. 22, n. 2, set. 2022
Edição Brazilian Symposium on
Fractional Calculus

A construction of nonstandard finite difference scheme to fractional differential system

Lislaine Cristina Cardoso
UEMS
lislaine.cardoso@uems.br

Fernando Luiz Pio dos Santos
UNESP - Universidade Estadual Paulista "Júlio de Mesquita Filho"
fernando.pio@unesp.br

Rubens de Figueiredo Camargo
Faculdade de Ciências
UNESP - Universidade Estadual Paulista "Júlio de Mesquita Filho"
rubens.camargo@unesp.br

Abstract

In this paper on the nonstandard finite difference technique is proposed to study numerically a fractional differential system. The Grünwald–Letnikov method is used to approximate the fractional derivatives. The model is given by fractional Caputo operator. The numerical results show that numerical solutions are in agree with analytical studies.

Keywords: Fractional calculus. Numerical methods. Fractional differential system

Este artigo está licenciado com uma Licença Creative Commons Attribution 4.0 International, podendo ser usado, distribuído e reproduzido, sem restrições, desde que o trabalho original seja devidamente citado.

1 Introduction

There is an expressive scientific advance in the development of theory and applications in Fractional Calculus, and several analytical techniques were proposed to solve fractional differential equations (CAMARGO; OLIVEIRA, 2015; CARDOSO; SANTOS; CAMARGO, 2018; SUN et al., 2018). However, finding explicit analytical solutions is not always an easy task. Thereby, the need arises to obtain approximate values for solving such equations, through numerical methods. In the literature, there are still relatively few numerical methods found for Fractional Calculus. In addition, depending on the definition of the operator Fractional used, the interactions can be extremely slow and expensive, due to the presence of persistent memory.

In this sense, numerical methods have been developed to solve fractional differential equations (FDEs). Since the structure of the finite difference method is simple, the Nonstandard finite difference schemes (NSFD) and Grünwald–Letnikov method (GL) are very popular methods for solving FDEs (BALEANU et al., 2021; ONGUN; ARSLAN; GARRAPPA, 2013; SHABBIR et al., 2019; ZIBAEI; NAMJOO, 2016).

The initial ideas of NSFD schemes came from the exact finite difference schemes (MICKENS; SMITH, 1990). Regarding the positivity, boundedness, and monotonicity of solutions, NSFD schemes have a better performance over the standard finite difference schemes, due to flexibility to construct an NSFD scheme that can preserve certain properties and structures, which are obeyed by the original equations. Besides these schemes, the discontinuous Galerkin methods for the fractional diffusion equations are proposed in (MUSTAPHA; MCLEAN, 2011). Delay fractional differential and various other related equations, Laguerre spectral methods have been used (KHADER, 2013). The Grünwald-Letnikov method for fractional differential equations is presented in (SCHERER et al., 2011). Because of the persisting memory of fractional-order operators, multi-step methods are used for FDEs. The Adams-Bashforth-Moulton method (ABM) is an example of the multi-step to solve FDEs (GARRAPPA, 2018; ZAYERNOURI; MATZAVINOS, 2016).

Motivated by the above discussion, we present the NSFD schemes to find the approximation of solutions of the fractional differential system and compare it with the fractional ABM and GL method.

The remainder of this paper is organized as follows. In Section 2, we recall some basic definitions concerning fractional-order operators. Section 3 is devoted to illustrating NSFD method for FDEs; Section 4, is devoted to showing the application and of test problems. Finally, Section 5 brings the concluding remakes.

2 Preliminaries Concepts

This section presents some definitions related to fractional calculus (CAMARGO; OLIVEIRA, 2015).

Definition 2.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a differential function and $\alpha \in \mathbb{C}$ such that $Re(\alpha) > 0$. The Riemann-Liouville fractional integral of order α of $f(t)$, $t \in \mathbb{R}$, denoted by $J^\alpha f(t)$, is defined as*

$$J^\alpha f(t) = \phi_\alpha(t) * f(t) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)\, d\tau, \tag{1}$$

where the symbol $*$ denotes the Laplace convolution and $\phi_\alpha(t)$ is the Gel'fand-Shilov function,

defined for $\alpha \notin \mathbb{Z}_-$, *as* $\phi_\alpha(t) = \begin{cases} \dfrac{t^{\alpha-1}}{\Gamma(\alpha)} & , \; if \; t \geq 0 \\ 0 & , \; if \; t < 0 \end{cases}$ *and* $\Gamma(\alpha)$ *is the Gamma function.*

There are several definitions for fractional-order derivatives, among which Caputo derivative and Riemann-Liouville derivative are the most frequently concerned in research (CAMARGO; OLIVEIRA, 2015). These definitions are presented as follows.

Definition 2.2 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a integrable function,* $\alpha \in \mathbb{C}$ *with* $Re(\alpha) > 0$, $\alpha \notin \mathbb{N}$ *and* $n - 1 < Re(\alpha) \leq n$, $t > \alpha$. *The Riemann-Liouville fractional derivative of order* α *of* $f(t)$ *is*

$$D^\alpha_{RL} f(t) = D^n[J^{n-\alpha} f(t)] = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau. \tag{2}$$

Definition 2.3 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be an differential function,* $\alpha \in \mathbb{C}$ *with* $Re(\alpha) > 0$ *and* m *the natural number, such that,* $m - 1 < Re(\alpha) \leq m$. *The Caputo fractional derivative of order* α *is defined as*

$$D^\alpha f(t) = J^{m-\alpha}\, D^m f(t) = \phi_{m-\alpha} * D^m f(t). \tag{3}$$

About the definition 2.3 we can observe:

- The usual derivative is recovered, since $J^0 f(t) = f(t)$, if $\alpha \in \mathbb{N}$.
- If $\alpha = m$ and $\beta = n$, with $n, m \in \mathbb{N}$, the classical result is obtained if *i.e.*, $D^\alpha t^\beta = \dfrac{t^{\beta-\alpha}\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}$, including the case where β is zero, *i.e.*, the derivative of the constant function is zero. This fact is one of the main differences between the fractional derivative of Caputo and Riemann-Liouville.

Next operator is very useful to obtain numerical solutions of fractional differential equations.

Definition 2.4 *The Grunwald-Letnikov operator (GL) is given by*

$$D^\alpha_{GL} f(t) = \lim_{h \to 0} h^{-\alpha} \sum_{j=0}^{[k]} w_j^{(\alpha)} f(t - jh) \qquad t \in [0, t_f], \tag{4}$$

where $0 < \alpha < 1$, $[k]$ *is the integer parts* $k = \frac{t-a}{h_k}$, *a and t are the reals limits of operator* D^α, *which denotes the fractional derivative,* $h_k > 0$ *is the step-size,* t_f *is the final time and* $w_j^{(\alpha)}$ *are the weights the coefficients in the power series expansion of* $(1-\xi)^\alpha$, i.e.,

$$(1-\xi)^\alpha = \sum_{j=0}^{\infty} w_j^{(\alpha)} \xi^j, \quad w_j^\alpha = \frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)}$$

and, from a practical point of view, they can be evaluated recursively by means of the following recurrence

$$w_0^{(\alpha)} = 1\,, \quad w_j^{(\alpha)} = \left(1 - \frac{1+\alpha}{j}\right) w_{j-1}^{(\alpha)}\,, \quad j = 1, 2, \ldots \tag{5}$$

$$w_0^{(\alpha-1)} = 1\,, \quad w_j^{(\alpha-1)} = \left(1 - \frac{\alpha}{j}\right) w_{j-1}^{(\alpha-1)}\,, \quad j = 1, 2, \ldots. \tag{6}$$

Lemma 2.1 *Let* $0 < \alpha < 1$ *and* $w_n^{(\alpha)}$, $w_n^{(\alpha-1)}$ *the coefficients in the GL operator. Then for any* $n = 1, 2, \ldots$ *we have* $-1 < w_n^{(\alpha)} < 0$ *e* $0 < w_n^{(\alpha-1)} < 1$.

Proof. Is an immediate consequence of the recursive relationship stated in (5) and (6).

Lemma 2.2 *Let* $w_n^{(\alpha)}$, *be the coefficient in the GL operator, then* $\sum_{j=0}^{n} w_j^{(\alpha)} = w_0^{(\alpha)} + w_1^{(\alpha)} + \ldots + w_n^{(\alpha)} = w_{n+1}^{(\alpha-1)}$.

Proof. Using finite induction to $n = 2$, we have $w_0^{(\alpha)} + w_1^{(\alpha)} + w_1^{(\alpha)} = 1 - \alpha + \frac{\alpha^2}{2} - \frac{\alpha}{2} = w_2^{(\alpha-1)}$. Suppose $n = k$, then

$$w_0^{(\alpha)} + w_1^{(\alpha)} + \ldots + w_k^{(\alpha)} = w_{k+1}^{(\alpha-1)}. \tag{7}$$

So let's show that it goes for $n = k + 1$,

$$w_0^{(\alpha)} + w_1^{(\alpha)} + \ldots + w_k^{(\alpha)} + w_{k+1}^{(\alpha)} = w_{k+1}^{(\alpha-1)}. \tag{8}$$

Using the equation (7), to prove the equation (8) is equivalent to prove

$$w_n^{(\alpha-1)} + w_{n+1}^{(\alpha)} = w_{n+1}^{(\alpha-1)}. \tag{9}$$

Then

$$\begin{aligned} w_{n+1}^{(\alpha)} &= w_{n+1}^{(\alpha-1)} - w_n^{(\alpha-1)} \\ &= \left(1 - \frac{\alpha}{n+1}\right) w_n^{(\alpha-1)} \\ &= w_n^{(\alpha-1)} \left(-\frac{\alpha}{n+1}\right). \end{aligned}$$

By property of Gamma function, we have

$$w_{n+1}^{(\alpha)} = \frac{\Gamma(n+1-\alpha)}{\Gamma(-\alpha)\Gamma(n+2)}, \tag{10}$$

and

$$w_n^{(\alpha-1)} \left(-\frac{\alpha}{n+1}\right) = \frac{\Gamma(n-\alpha+1)}{\Gamma(-\alpha+1)\Gamma(n+1)} \frac{(-\alpha)}{(n+1)} = \frac{n+1-\alpha}{\Gamma(-\alpha)\Gamma(n+2)}. \tag{11}$$

By equations (10) and (11) we can write $w_n^{(\alpha-1)} + w_{n+1}^{(\alpha)} = w_{n+1}^{(\alpha-1)}$. Then

$$w_0^{\alpha} + w_1^{\alpha} + w_2^{\alpha} + \ldots + w_n^{\alpha} = w_n^{(\alpha-1)}.$$

3 NSFD for fractional order system

Let be a nonlinear fractional differential equation,

$$D^{\alpha} y(t) = f(t, y(t), \lambda), \ 0 < \alpha \leq 1 \tag{12}$$

λ is a vector parameter. For the construction of the scheme, the left-hand derivative of the equation (12) is replaced, using the derivative of GL, by the discrete representation in the form

$$D^{\alpha} y(t) \approx \frac{1}{\phi(h, \lambda)} \sum_{j=0}^{n} w_j^{(\alpha)} (y_{n-j} - y_0), \tag{13}$$

where a_n is an approximation of $y(t_n)$, $h > 0$ is the time step and $\phi(h, \lambda)$ is the denominator function. The nonlinear term of equation (12) is given by nonlocal representation

$$F(t_n, y_n, y_{n-1}, ..., \lambda) \ , \quad n = 1, 2, 3, ... \tag{14}$$

Thus, by the equations (13) and (14), we can rewrite the equation (12) as

$$\frac{1}{\phi(h, \lambda)} \sum_{j=0}^{n} w_j^{(\alpha)} (y_{n-j} - y_0) = F(t_n, y_n, y_{n-1}, ..., \lambda).$$

The discretized form for the equation (12), as follows

$$y_n = y_0 w_n^{(\alpha-1)} - \sum_{j=1}^{n-1} w_j^{(\alpha)} y_{n-j} + \phi(h, \lambda) F(t_n, y(t_n)). \tag{15}$$

The denominator function $\phi(h, \lambda)$ is a function of h, and must satisfy the consistency condition,

$$\phi(h, \lambda) = h + O(h^p), \, p > \alpha, \, h \to 0,$$

which ensures the convergence of the discrete approximation (13) to the associated continuous derivative as h tends to zero, where $O(h^p)$ is the truncate error.

Examples of denominator functions that satisfy the above condition are $\phi_1 = \dfrac{1 - e^{-h^\alpha \lambda}}{\lambda}$, $\phi_2 = \left[\dfrac{1 - e^{-h\lambda}}{\lambda}\right]^\alpha$, $\phi_3 = h^\alpha$, $\phi_4 = \dfrac{\sin(h^\alpha \lambda)}{\lambda}$; The influence of the denominator function can be seen in (CARDOSO; SANTOS; CAMARGO, 2021). When $\phi(h, \lambda) = h$, in the equation (13) the NSFD scheme reduces to the GL scheme (SCHERER et al., 2011).

4 Results

To illustrate the application of the numerical method we use the fractional-order Hepatitis B model (CARDOSO; SANTOS; CAMARGO, 2018). This model describes the viral infection caused by the HBV virus that attacks the liver cells and it can cause chronic infection and evolve to cirrhosis or liver cancer. The fractional-order model is given by

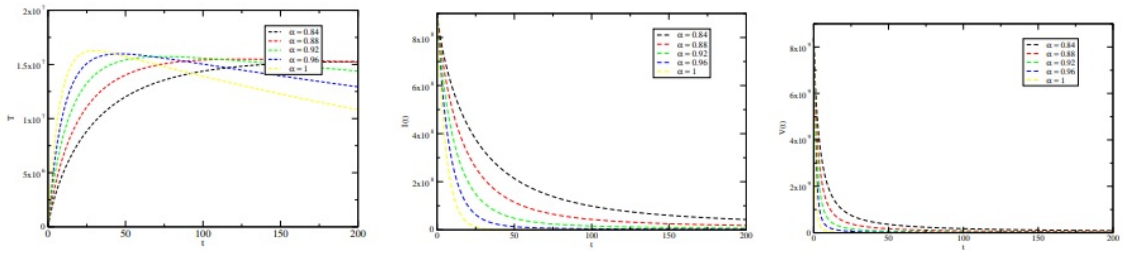
$$\begin{cases} D^\alpha T(t) &= s - dT(t) - \beta V(t)T(t) + \rho I(t) \\ D^\alpha I(t) &= \beta V(t)T(t) - \delta' I(t) - \rho I(t) \\ D^\alpha V(t) &= pI(t) - cV(t). \end{cases} \tag{16}$$

where D^α is Caputo fractional derivative, $\alpha \in (0, 1]$ is the fractional order, and the biologicals parameters are:

- $T(t)$, $I(t)$, $V(t)$: uninfected cells, infected cell and free virus;
- s is the rate of production of new uninfected cells;
- d is the death rate of uninfected cells;
- β rate of infection of new uninfected cells;

- ρ the rate of cure;
- δ' is the death rate of infected cells;
- $\delta = \delta' + \rho$ the net loss rate of infected cells;
- p is the rate of production of virus per infected cell;
- c free virus clearance rate.

To performe the simulations the initial conditions are $T(0) = 2.3 \times 10^6$, $I(0) = 9.1 \times 10^8$, $V(0) = 8.5 \times 10^9$ and the values of the biologicals parameters $s = 5\ day^{-1}$, $d = 0.01\ day^{-1}$, $\beta = 0.0012\ day^{-1}$, $\delta = 0.5847$, $c = 2.8058$, $p = 6.0255$, and $\rho = 0.01\ day^{-1}$.



(a) $T(t)$ (b) $I(t)$ (c) $V(t)$

Figure 1: State variables of the fractional-order model in the time-domain $\alpha = 0.84, 0.88, 0.92, 0.96, 1$. The step size is $h = 0.025$.

In Figure 1, we can see the numerical solution of model (16) with several values of the fractional order, α. As can be seen, following the course of the disease, the system evolves to a state of equilibrium. The smallest values to α imply slower convergence to the equilibrium state. These numerical results confirmed the analytic prediction, given in (CARDOSO; SANTOS; CAMARGO, 2018).

Table (1) shows the maximum error value $\xi_x(h) = \max\{|\overline{x}_n - x_n^*|\}$ committed to approximate the numerical solution of the model, for several values of the fractional exponent, α, different values for h and $n = 300$.

Table 1: Global error by NSFD

α \ h	10^{-2}	10^{-3}	10^{-4}	10^{-5}
0.2	6.9×10^{-2}	4.5×10^{-2}	3.2×10^{-3}	1.9×10^{-3}
0.4	4.5×10^{-2}	3.9×10^{-2}	3.6×10^{-3}	9.5×10^{-4}
0.6	3.9×10^{-2}	9.2×10^{-3}	6.6×10^{-4}	5.1×10^{-4}
0.8	2.1×10^{-2}	7.7×10^{-3}	5.1×10^{-4}	4.6×10^{-4}
0.9	9.8×10^{-3}	5.5×10^{-3}	3.5×10^{-4}	4.4×10^{-4}
0.95	7.1×10^{-3}	4.9×10^{-3}	3.3×10^{-4}	2.9×10^{-4}
0.97	6.1×10^{-3}	9.3×10^{-4}	2.2×10^{-4}	2.1×10^{-4}
1	4.3×10^{-3}	8.1×10^{-4}	4.3×10^{-5}	1.1×10^{-5}

Table (1) shows that, as we increase the value of the fractional exponent and refine the value of h, there is a reduction in error between approximations for most cases. The smaller error obtained

between the approximations is for $\alpha = 1$ and $h = 10^{-5}$, in the which $\xi_x(h) = 1.1 \times 10^{-5}$. This implies that the solution numerical solution is very close to the exact solution.

To analyze the order of convergence (ψ) of the NSFD method, a the estimate was calculated according to the relationship

$$\psi_\alpha = \log_2 \left(\frac{\xi(2^\alpha h^\alpha)}{\xi(h^\alpha)} \right),$$

where ξ is the global error corresponding to step h and with relation to the fractional exponent α. The Table 2 shows the order of convergence of the method NSFD.

Table 2: Convergence

α \ h	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
0.2	0.1022	0.1547	0.1923	0.2064	0.2807
0.4	0.3571	0.3741	0.4012	0.4074	0.4897
0.6	0.5490	0.5803	0.6037	0.6743	0.7216
0.8	0.8495	0.8623	0.8607	0.8967	0.9023
0.9	0.8793	0.9043	0.9175	0.9672	1.1247
0.95	0.9108	0.9239	0.9515	0.9612	1.2278
0.97	0.9511	0.9543	0.9575	0.9772	1.0641
1	0.9873	0.9903	0.9995	1.0671	1.5239

In Table (2) we can see that as α decreases, there is a decrease in the convergence of the method. Furthermore, we can observe that the NSFD method is convergent with the order of convergence equal to α, which is in agreement with the analytical results.

5 Conclusions

In this paper, we presented an NSFD scheme as a tool to solve numerically a fractional differential system. This scheme was satisfactory because the solution obtained are in agreement with the analytical results.

References

BALEANU, D. et al. A nonstandard finite difference scheme for the modeling and nonidentical synchronization of a novel fractional chaotic system. **Advances in Difference Equations**, Berlim, v. 2021, n. 1, p. 1–19, 2021.

CAMARGO, R. F.; OLIVEIRA, E. C. **Cálculo fracionário**. São Paulo: Editora Livraria da Fısica, 2015.

CARDOSO, L. C.; SANTOS, F. L. P.; CAMARGO, R. F. Analysis of fractional-order models for hepatitis b. **Computational and Applied Mathematics**, Rio de Janeiro, v. 37, n. 4, p. 4570–4586, 2018.

CARDOSO, L. C.; SANTOS, F. L. P.; CAMARGO, R. F. Método de diferenças finitas não clássico: influência da função denominador na solução numérica de um modelo de ordem fracionária. **Proceeding Series of the Brazilian Society of Computational and Applied Mathematics**, Rio de Janeiro, v. 8, n. 1, 2021.

GARRAPPA, R. Numerical solution of fractional differential equations: a survey and a software tutorial. **Mathematics**, Switzerland, v. 6, n. 2, p. 16, 2018.

KHADER, M. The use of generalized laguerre polynomials in spectral methods for solving fractional delay differential equations. **Journal of Computational and Nonlinear Dynamics**, Michigan, v. 8, n. 4, 2013.

MICKENS, R. E.; SMITH, A. Finite-difference models of ordinary differential equations: influence of denominator functions. **Journal of the Franklin Institute**, Amsterdam, v. 327, n. 1, p. 143–149, 1990.

MUSTAPHA, K.; MCLEAN, W. Piecewise-linear, discontinuous galerkin method for a fractional diffusion equation. **Numerical Algorithms**, Amsterdam, v. 56, n. 2, p. 159–184, 2011.

ONGUN, M. Y.; ARSLAN, D.; GARRAPPA, R. Nonstandard finite difference schemes for a fractional-order brusselator system. **Advances in Difference Equations**, London, v. 2013, n. 1, p. 1–13, 2013.

SCHERER, R. et al. The grünwald–letnikov method for fractional differential equations. **Computers & Mathematics with Applications**, Amsterdam, v. 62, n. 3, p. 902–917, 2011.

SHABBIR, M. S. et al. A dynamically consistent nonstandard finite difference scheme for a predator–prey model. **Advances in Difference Equations**, London, v. 381, n. 1, p. 1–17, 2019.

SUN, H. et al. A new collection of real world applications of fractional calculus in science and engineering. **Communications in Nonlinear Science and Numerical Simulation**, Amsterdam, v. 64, p. 213–231, 2018.

ZAYERNOURI, M.; MATZAVINOS, A. Fractional adams–bashforth/moulton methods: an application to the fractional keller–segel chemotaxis system. **Journal of Computational Physics**, Amsterdam, v. 317, p. 1–14, 2016.

ZIBAEI, S.; NAMJOO, M. A nonstandard finite difference scheme for solving three-species food chain with fractional-order lotka-volterra model. **Iranian Journal of Numerical Analysis and Optimization**, Mashhad, v. 6, n. 1, p. 53–79, 2016.