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The Basset force and the persistence of hydrodynamic memory

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Abstract

The motion of a particle through a viscous fluid generates vorticity, which in turn induces a delayed viscous force on the particle that can dramatically modify its motion at later times. This viscous self-interaction is known as the Basset history force, which is responsible for the well-known phenomenon of hydrodynamic memory. Although the Basset problem has been studied for well over a century, there has recently been growing interest in the effects of hydrodynamic memory upon the driven transport of micro- and nanoparticles. Such transport typically takes place at very low Reynolds number, where the non-uniform motion of a spherical particle is well described by the Basset-Boussinesq-Oseen (BBO) equation. Recently, numerical simulations of nonlinear particle diffusion and transport have revealed that hydrodynamic memory can have rather striking dynamical consequences, which is ultimately caused by a nontrivial interplay between the Basset history force and external forcing. Here, we examine this interplay in the BBO equation from an analytical standpoint for general time-dependent forcing. In particular, we use standard Laplace transform techniques to derive an explicit expression for the hydrodynamic coupling force, which we then use to demonstrate the indefinite persistence of memory effects—previously observed in numerical simulations—under nonequilibrium forcing conditions.

Keywords: Basset force. Riemann-Liouville fractional derivative. BBO equation. Hydrodynamic memory. micro-particles.

1 Introduction

The study of viscous particle motion was initiated by Stokes (STOKES, 1851), who determined the force acting on a small fixed particle that is subjected to a uniform fluid velocity at a low Reynolds number. Boussinesq (BOUSSINESQ, 1885) and Basset (BASSET, 1888) independently extended the work of Stokes by considering the case where a spherical particle accelerates through the fluid due to a constant gravitational force but still neglecting non-linear effects. They found that the hydrodynamic force **F** acting on a spherical particle undergoing arbitrary time-dependent motion in an otherwise quiescent fluid is

$$\mathbf{F}(t) = -6\pi\mu R\,\mathbf{u}(t) - \frac{2}{3}\pi\rho R^3\frac{d\mathbf{u}(t)}{dt} - 6\pi\mu R\,(\frac{R^2}{\pi\nu})^{\frac{1}{2}}\int_{-\infty}^{t}\frac{1}{\sqrt{t-\tau}}\frac{d\mathbf{u}(\tau)}{d\tau}\,d\tau, \tag{1}$$

where ρ is the density, μ and $\nu = \frac{\mu}{\rho}$, are the dynamic and kinematic viscosities respectively, $\mathbf{u}(t)$ is the particle velocity, R is the particle radius and t represents the time. The first term is the pseudo-steady Stokes drag. The second term, a purely inertial contribution, is the so-called added mass term. It represents the additional mass the particle appears to have due to the resistance to the acceleration of the surrounding fluid. The third term is the Basset memory integral, which depends on the history of particle motion. It is a combination of both viscous and inertial contributions to the force in that it depends on both the viscosity of the fluid and the acceleration of the particle. Oseen (OSEEN, 1927) extended the work of Boussinesq and Basset, including the effects of higher Reynolds number on the equations. Due to the original contributions of Boussinesq, Basset, and Oseen. the particle equation of motion with a constant forcing (the gravity term) is sometimes referred to as the BBO equation.

Due to the considerable computational challenges, the Basset history force is often neglected, very few attempts (GUSEVA; FEUDEL; TÉL, 2013; PRASATH; VASAN; GOVINDARAJAN, 2019; TOEGEL; LUTHER; LOHSE, 2006; GARBIN et al., 2009) have been made to try to analyze the effect of the history force on the dynamics of the particle. The general problem of the description of the unsteady motion of a particle falling through a viscous fluid and due to its relative acceleration with respect to the fluid itself is now referred to as the Basset problem.

A mathematical speculation first proposed by Mainardi et al. (MAINARDI et al., 1995) and based on a generalized Basset force, which is expressed in terms of a fractional derivative in Caputo's sense of any order α ranging in the interval $0 < \alpha < 1$ has proved very fertile ground for basing the study of the Basset problem. They (MAINARDI et al., 1995) expected that their mathematical speculation could provide a phenomenological insight for the experimental data. Several articles from the same group elaborated this inspirational insight (GORENFLO; MAINARDI, 2008; MAINARDI; PIRONI; TAMPIERI, 1995; GORENFLO; MAINARDI, 2019). Recently experimental observations of power-law temporal response for spheres in the transient regime of low Reynolds number flow under the effect of gravity were provided (BUONOCORE; SEN; SEMPERLOTTI, 2019). This fascinating result is consistent with the generalized fractional order Basset force proposed by Mainardi et al. (MAINARDI et al., 1995). Almost at the same time, Seyler and Pressé (SEYLER; PRESSÉ, 2019) studying the BBO equation describing the motion of a driven single-particle discovered numerically that memory effects persist indefinitely under rather general driving conditions. Thus, showing that neglecting the history force can lead to qualitatively incorrect particle transport under general non-equilibrium conditions.

The purpose of this note is to explore the significance of the Basset history force in the dynamics of a spherical particle in a low Reynolds number viscous flow. With this objective in mind, we revisit the generalized Basset problem including a time-dependent external applied force into the BBO equation. Using the methodology of the Laplace transform, we obtain the solution of the BBO equation in terms of the convolution between the time-rate-of-change of the external driven force and a function $M(t)$, the latter of which decays algebraically. We study in detail two particular cases of driving forces: (i) a constant pulse force, and (ii) a sinusoidal force. The solution shows the existence of a coupling between the generalized Basset force and the external force, responsible for the persistence of memory effects. The plan of the present work is as follows. First, we consider the dynamics of a particle with the same density as the surrounded fluid, in the presence of an arbitrary time dependent driven force, we study in detail two particular examples, a driven impulse force, and a sinusoidal force. Then, we consider the non-neutral buoyancy case, the density of the particle greater than the density of the fluid. We study the dynamics of a particle for two particular examples, a driven impulse force, and a sinusoidal force. Finally, we draw our conclusions.

2 The Generalized Basset Problem revisited

The generalized Basset problem was introduced by Mainardi et al. (MAINARDI et al., 1995) and extended in several articles by Mainardi and co-workers (GORENFLO; MAINARDI, 2008; MAINARDI; PIRONI; TAMPIERI, 1995; GORENFLO; MAINARDI, 2019). In this section we are going to consider the generalized Basset problem including a time-dependent driven force. First, we may consider the case in which the densities of the particle ρ_p, and the fluid are equal numerically studied by (SEYLER; PRESSÉ, 2019), ρ_f, and then we may consider the non-neutral buoyancy case in which $\rho_p > \rho_f$. In both cases, we study two particular situations; an impulse force and a sinusoidal time-dependent driven force.

2.1 Neutral buoyancy.

The nonuniform motion of a spherical particle in an incompressible fluid in the limit of low Reynolds number is described by the BBO (MAINARDI et al., 1995; GORENFLO; MAINARDI, 2008; MAINARDI; PIRONI; TAMPIERI, 1995), in the case in which there is an external time-dependent applied force, the governing equation can be written in dimensionless variables as follows

$$\frac{du}{dt} + a\,{}_0^C D_t^{\alpha} u + u = F(t), \qquad a = \sqrt{\frac{9\rho_f}{2\rho_p + \rho_f}}\,\Gamma^{-1}(1-\alpha), \tag{2}$$

where $u = u(t)$ is the particle's velocity, $\Gamma(\alpha)$ the gamma function, $F(t)$ is the external driving force, and ${}_0^{RL}D_t^{\alpha}$ is the Caputo's fractional derivative of order α, defined by

$$ {}_0^C D_t^{k-\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-\tau)^{1-\alpha}} \frac{d^k}{dt^k}\, u\, d\tau, \tag{3}$$

where $k = 1, 2, 3, ..,$ and $k - 1 < \alpha < k$, we consider the case n=1. It is worth noticing that Eq. (2) was considered in (KILBAS; SRIVASTAVA; TRUJILLO, 2006) (see example 5.21 eq. (5.3.79), the solution even though very elegant, a bit cumbersome for our purpose in the present work. Seyler and Pressé (SEYLER; PRESSÉ, 2019) considered the above Equation. They applied a Markovian embedding technique. This method allows us to transform a history-dependent integro differential equation into a numerically tractable system of equations that are local in time. In contrast to

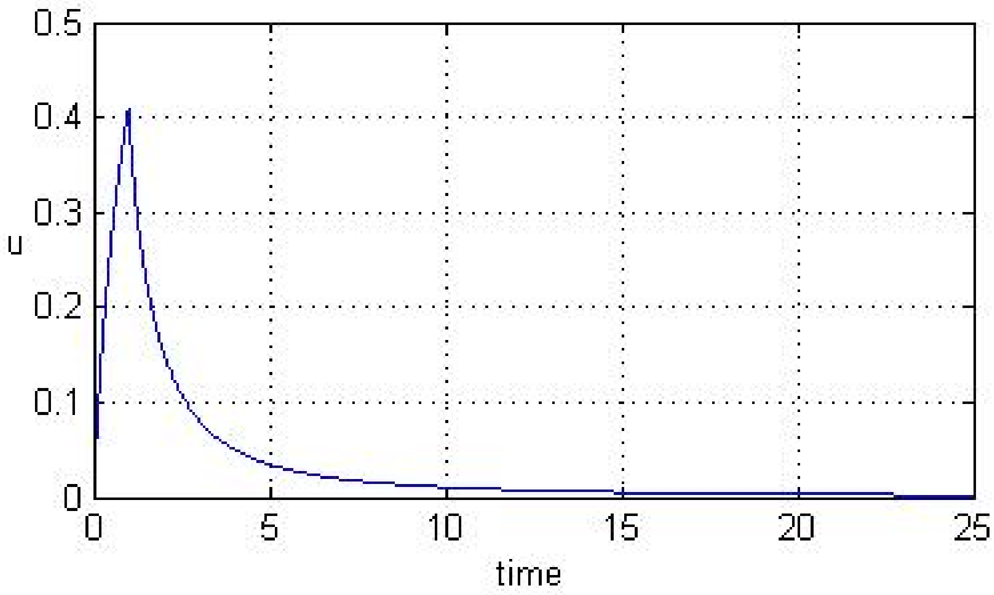

Figure 1: The particle's velocity as a function of time for a applied step pulse of time $\Delta t = 1$.

(SEYLER; PRESSÉ, 2019), we approach Eq. (2) with initial condition $u_0 = 0$ using analytical techniques. Applying the Laplace transform to (2), we get

$$L\{u\} = \frac{1}{(1 + s + a\, s^{\alpha})} F(s). \tag{4}$$

where following Mainardi et al. (MAINARDI et al., 1995)

$$N(s) = \frac{1}{(1 + s + a\, s^{\alpha})} = 1 - s\, M(s), \qquad M(s) = \frac{(1 + a\, s^{\alpha-1})}{(1 + s + a\, s^{\alpha})} \tag{5}$$

thus Eq. (4) can be written as

$$L\{u\} = \big(1 - s\, M(s)\big)\, F(s), \tag{6}$$

taking the inverse Laplace transform gives us

$$u(t) = F(t) + F_c(t), \tag{7}$$

where

$$F_c(t) = -L^{-1}\{s\, M(s)\, F(s)\} \tag{8}$$

represents the coupling between the Basset force and the time-dependent driven force. In (SEYLER; PRESSÉ, 2019), they compared the behavior of a particle in the presence of Basset and Stokes forces (BBO beads) with the behaviour of the particle in the presence of the Stokes force (Stokes beads) without Basset force. In the case of Stokes beads Eq. (2) becomes a relaxation differential equation see (GORENFLO; MAINARDI, 2019).

Let us now turn our attention to the BBO beads, in order to bolster physical intuition let us study the case of a driven impulse force $F(t) = 1 - H(t - t_0)$ where H(t) is the Heaviside step function, this case was studied in (SEYLER; PRESSÉ, 2019), substituting into (8), and using the convolution theorem,

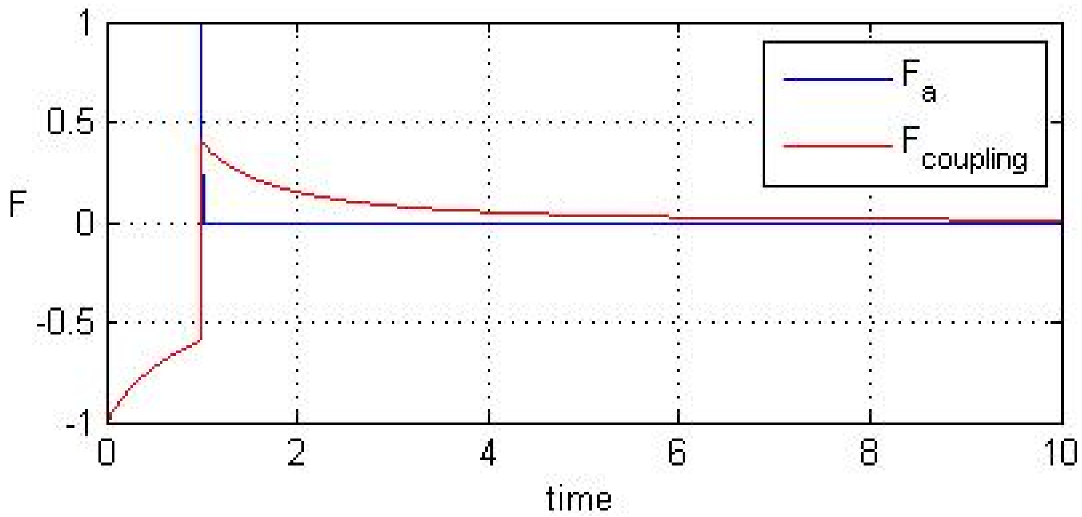

Figure 2: The plot of the driven step pulse F_a, with $\Delta t = 1$, together with the coupling force given by $F_c = -L^{-1}\{s\, M(s)\, F(s)\}$.

$$u(t) = 1 - H(t - t_0) + \int_0^t M(\tau)\, \delta(t - \tau)\, d\tau - \int_0^t M(\tau)\, \delta(t - t_0 - \tau)\, d\tau \tag{9}$$

integrating the third and fourth terms of Eq. (9) gives us

$$u(t) = 1 - H(t - t_0) + M(t - t_0) - M(t) \tag{10}$$

now, using the result $M(0) = 1$, and $M(t - t_0) > M(t)$ (MAINARDI, 2012), We find that the memory effects persist indefinitely, it is worth noticing that, Seyler and Pressé (SEYLER; PRESSÉ, 2019), were able to compute numerically this fascinating effect of the Basset history force. Fig. 1 shows the plot of the velocity as a function of time for $\alpha = 0.5$, Fig. 2 shows the plot of an impulse driving force with a time duration of $\Delta t = 1$ and the coupling force we see that the magnitude of the coupling force is non-zero during a very long interval of time, this is the memory persistence numerically computed by Seyler and Pressé (SEYLER; PRESSÉ, 2019). Figure 3 shows a plot of a double step driving force and the coupling force, the separation between steps is $\Delta t = 4$, we see that the magnitude of the coupling force increases.

Let us consider the case of a driven sinusoidal force of amplitude b and frequency ω, $F(t) = b\ \sin(\omega\, t)$, the particle's velocity will be (using (7)-(8))

$$u(t) = b\ \sin(\omega\, t) - b\, L^{-1}\{\frac{(1 + a\, s^{\alpha-1})}{(1 + s + a\, s^{\alpha})}\, (\frac{\omega}{\omega^2 + s^2})\}. \tag{11}$$

applying the convolution theorem, we get

$$u(t) = b\ \sin(\omega\, t) - b\ \omega \int_0^t M(\tau)\ \sin\left(\omega\,(t - \tau)\right) d\tau \tag{12}$$

using the fact that $M(t)$ is a monotonic decreasing function with $M(0) = 1$ see (MAINARDI, 2012), we may see that the second term on the right hand of Eq. (12) decreases as time progresses. thus, the amplitude of the particle's oscillations increases as time progresses this result was obtained also in (PRASATH; VASAN; GOVINDARAJAN, 2019) for the particular case $\alpha = \frac{1}{2}$. We may conclude

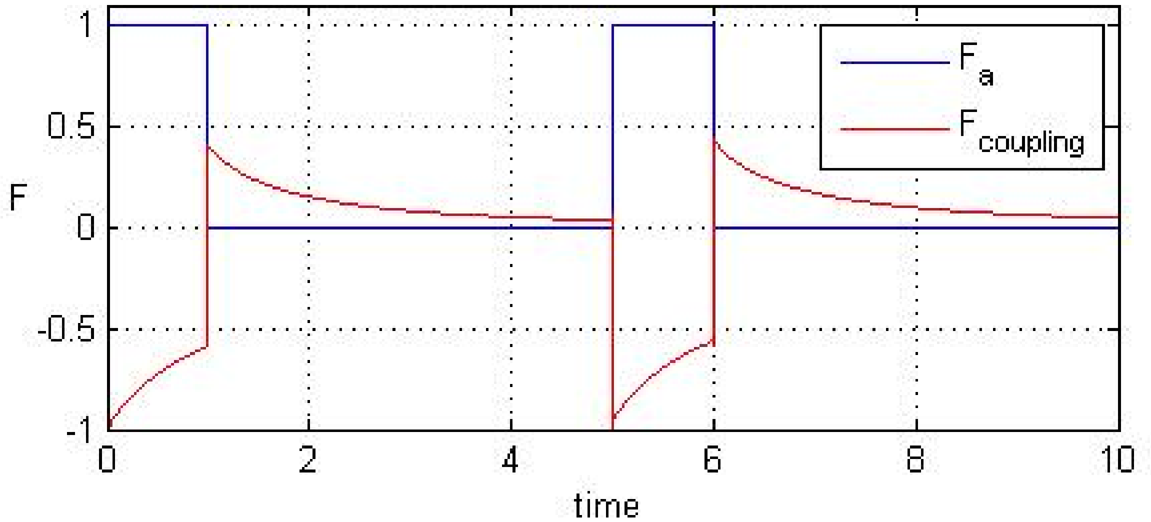

Figure 3: The plot of the double step pulse double F_a, with $\Delta t = 1$, together with the particle's response, the coupling is given by $F_c = -L^{-1}\{s\, M(s)\, F(s)\}$

that the particle's velocity oscillate with the same frequency of the driven force and with a time dependent increasing amplitude. Thus, the Basset history force is not just an additional drag force.

In the next sections we will consider the case in which the density of the particle ρ_p is greater than the density of the fluid ρ_f, again we study two rather interesting external driven forces: a impulse force $F(t) = 1 - H(t - t_0)$ see for example (SEYLER; PRESSÉ, 2019), and a sinusoidal force $F(t) = b\ \sin(\omega\, t)$.

2.2 Non-neutral buoyancy $\rho_p > \rho_f$.

In this section we may consider the case in which the particle possesses a density greater than the fluid and the particle is in the presence of a driven time dependent force. In this case the BBO equation can be written as follows,

$$\frac{du}{dt} + a\, {}_0^C D_t^\alpha u + u = 1 + F(t), \quad u(0) = u_0, \quad 0 < \alpha < 1. \tag{13}$$

This model was introduced by Mainardi et al (MAINARDI et al., 1995). without time dependent driven force. Repeating the procedure of the Laplace transform method to Eq. (13) we will obtain,

$$u(t) = 1 + (u_0 - 1)\, M(t) + F(t) + F_c(t), \tag{14}$$

where $F_c(t)$ represents the interaction between the external driven time dependent force and the Basset history force, where $F_c(t)$ represents the interaction between the external driven time dependent force and the different forces acting upon the particle; the Basset history force, Stokes and inertia forces, the coupling is given by (the inverse Laplace transform of $-s\, M(s)\, F(s)$),

$$F_c(t) = -L^{-1}\{s\, M(s)\, F(s)\} \tag{15}$$

the term $M(t)$ is given by

$$M(t) = L^{-1}\{\frac{(1 + a\, s^{\alpha-1})}{(1 + s + a\, s^\alpha)}\},$$

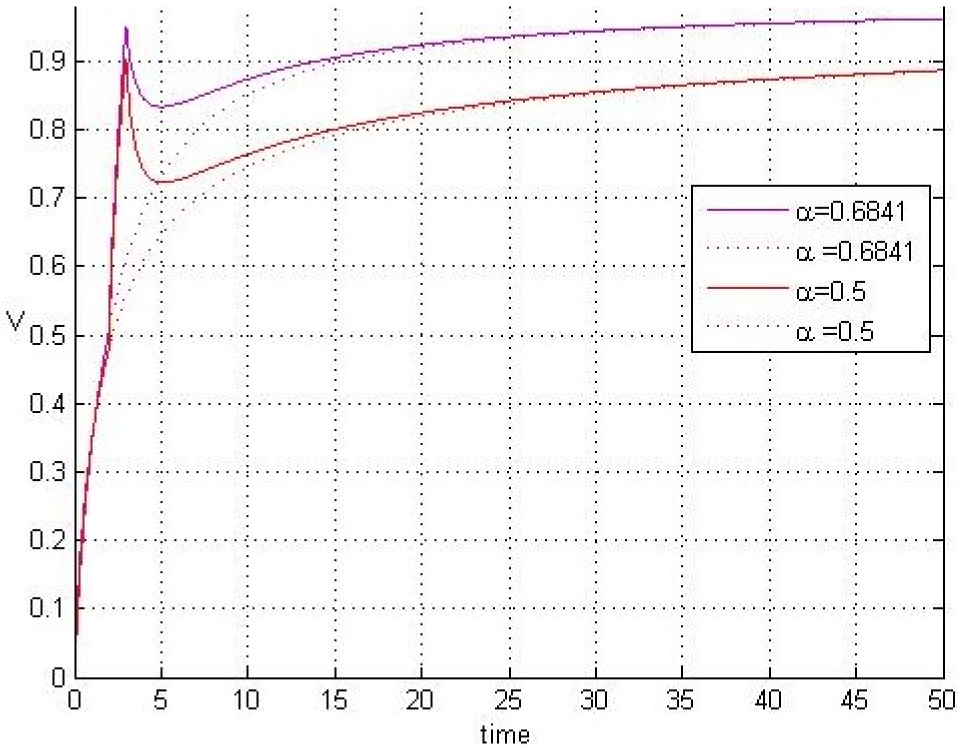

Figure 4: The plot of the velocity as a function of time for $\alpha = 0.6841$ and $\alpha = 0.5$ with (solid line) and without (dotted line) external force.

$$M(t) = \int_0^\infty e^{-s\,t}\, K(s)\, ds, \tag{16}$$

where $K(s)$ is given by see (MAINARDI, 2012)

$$K(s) = \frac{1}{\pi}\, \frac{a\, s^{\alpha-1}\, \sin(\alpha\, \pi)}{((1-s)^2 + a^2 s^{2\,\alpha} + 2(1-s)\, a\, s^{\alpha}\, \cos(\alpha\, \pi))} > 0.$$

If we define $\hat{F}(s) = s\, F(s)$ and using the convolution theorem, we finally obtain an explicit relation for the coupling F_c between the external applied force and the Basset force,

$$F_c(t) = -\int_0^t M(\tau)\, \hat{F}(t-\tau)\, d\tau. \tag{17}$$

Let us consider a particle driven by an impulse force $F(t) = 1 - H(t - t_0)$, using Eq. (14) and applying the convolution theorem the velocity will be

$$u(t) = 1 + (u_0 - 1)\, M(t) + 1 - H(t - t_0) + M(t) - M(t - t_0)\,, \tag{18}$$

Now, in the absence of the driven force the particle will fall under the action of gravity and will tend to a constant terminal velocity. Fig. 4 shows the plot of the velocity as a function of time for the cases $\alpha = 0.5$, and 0.6841, the dotted line shows the behaviour without driven force while the solid line shows the behaviour in the presence of a step external pulse. We can see that the effect of the applied force is to produce a transient velocity overshoot. The velocity overshoot is a phenomenon occurring in the free fall of spheres in a viscoelastic fluids (BECKER et al., 1994).

Let us consider a smooth external driven force, in the case in which the driven force is sinusoidal, $F(t) = b\ \sin(\omega\, t)$, the particle's velocity will be after applying again the convolution theorem given by

$$u(t) = 1 + (u_0 - 1)\, M(t) + b\ \sin(\omega\, t) - b\ \omega \int_0^t M(\tau)\ \sin\left(\omega\, (t-\tau)\right) d\tau. \tag{19}$$

We may see that the fourth term on the right hand of Eq. (19) decreases as time progresses. Another fascinating result is that, the amplitude of the particle oscillations increases as time progresses.

3 Conclusion

In this work, we revisit the generalized Basset problem proposed by Mainardi et al. (MAINARDI et al., 1995) including the effects of a time-dependent external applied force. From this analysis, we have identified a coupling term F_c, which can be interpreted as a convolution between the time-rate-of-change of the force and the function M(t), the latter of which decays algebraically. The coupling being responsible for the persistence of memory effects recently numerically discovered in (SEYLER; PRESSÉ, 2019). Thus, we see that the history force possesses a dual role it absorbs some of the input energy as observed by (SEYLER; PRESSÉ, 2019) and also acts as a damping force, the same behaviour of the Basset force was reported in (PRASATH; VASAN; GOVINDARAJAN, 2019) in the special case $\alpha = \frac{1}{2}$. It is worth noticing that Berman and Cederbaum (BERMAN; CEDERBAUM, 2018) found that the solution of the fractional driven oscillator decay algebraically and to possess a finite number of zeros. We think that the case of sinusoidal driving force might also be quite helpful for problems involving driven particle transport (with both space- and time-dependent forcing), such as, viscoelastic (SCHIEBER; CÓRDOBA; INDEI, 2013) and compressible fluids (TOEGEL; LUTHER; LOHSE, 2006; GARBIN et al., 2009; PARMAR; HASELBACHER; BALACHANDAR, 2011)).

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