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Numerical approximation of the integral generalized k-fractional Hilfer type

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Abstract

Recent studies show the importance of the fractional calculation in the modeling of problems in physics and engineering among others. These dynamic systems generally need numerical approximations to estimates of your solutions. This work presents an approximation numerical for k-fractional integral using as techniques of approximation the rule of the rectangle and the rule of the trapezoid.

Keywords: k-fractional integral, rule of the rectangle, rule of the trapezoid.

1 Introduction

The integral (k,ρ)-fractional, was proposed by Sarikaya in (SARIKAYA et al., 2016) and studied in (OLIVEIRA, 2018) which is a generalization of fractional integrals k—Riemann-Liouville and k—Hadamard when $\rho \to 1$ and $\rho \to 0^+$ respectively, furthermore when $k \to 1$ the generalized fractional integral is recovered proposed by Katugampola in (KATUGAMPOLA, 2014). Many applications has been investigated considering the fractional calculus (FC). However, the nonlinear problems need the numerical approach to better understanding to study of problem. As alternative, the numerical method to fractional k- Riemann- Liouvile integral are present, using two method to discretization of the fractional operator.

Definition 1.1 *Let $-\infty \le a < b < \infty$, $k,\gamma > 0$ y $\varphi \in L(a,b)$, defines the integral k-fractional left by*

$$\left({}_k\mathcal{J}^{\gamma}_{a^+}\varphi\right)(x) = \frac{1}{k\Gamma_k(\gamma)}\int_a^x \frac{\varphi(t)}{(x-t)^{1-\frac{\gamma}{k}}}dt, \quad x > a, \tag{1}$$

where the function is the known Γ_k defined in (DIAZ; PARIGUAN, 2004).

Definition 1.2 *Let $k > 0$, $z \in \mathbb{C}$ con $Re(z) > 0$ y $z \neq kn$ with $n \in \mathbb{Z}^-$, the function Γ_k is given by the integral*

$$\Gamma_k(z) = \int_0^\infty t^{z-1}e^{-\frac{t^k}{k}}dt. \tag{2}$$

The following results are useful for our purposes.

Proposition 1.3 *Let k, z as in definition 1.2 write the function Γ_k in terms of the function Γ of the form*

$$\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right). \tag{3}$$

The integral k-fractional to the left fulfills the following semigroup property (SARIKAYA et al., 2016), (OLIVEIRA, 2018).

Theorem 1.4 *Let $\gamma_1, \gamma_2 > 0$ y $\varphi \in L_p(a,b)$, then*

$$\left({}_k\mathcal{J}^{\gamma_1}_{a+}\,{}_k\mathcal{J}^{\gamma_2}_{a+}\varphi\right)(x) = \left({}_k\mathcal{J}^{\gamma_1+\gamma_2}_{a+}\varphi\right)(x). \tag{4}$$

Following what is stated in (BALEANU et al., 2012) the operator ${}^{\rho}_{k}\mathcal{J}^{\gamma}_{a^+}$ can be interpreted as an integral operator fifty in some proper sense, for this reason it is quite natural to use approximation methods based on in quadrature theory, that is, numerical integration theory, such as the rules Newton-Cotes quadrature, which we will modify a bit given the fractional nature of our operator.

In practice, constant jump interpolation polynomials are used to construct a function $\widetilde{\varphi}$ from the function φ on which the integration is carried out, we will carry out the same idea but with polynomial functions in the variable t.

Let us consider a partition of the interval $[a,x]$ where the nodes are given by $x_j = a + jh$, $j = 0,1,...,N$ where we have the jump $h = \frac{x-a}{N}$.

2 Rectangle rule for numerical approximation of the operator integral

Initially, we are going to approximate φ by piecewise function, which in each interval $[x_j, x_{j+1}]$ is a polynomial of degree 0, so the function constructed is of the form:

$$\widetilde{\varphi}(t) = \varphi(x_j) \quad si \quad x_j \leq t \leq x_{j+1} \tag{5}$$

in this way to numerically approximate $\left({}_k\mathcal{J}^{\gamma}_{a^+}\varphi\right)(x)$ y and we do it through $\left({}_k\mathcal{J}^{\gamma}_{a^+}\widetilde{\varphi}\right)(x)$ which we will denote by

$$_h\left({}_k\mathcal{J}^{\gamma}_{a^+}\varphi\right)_{Re}(x) = \left({}_k\mathcal{J}^{\gamma}_{a^+}\widetilde{\varphi}\right)(x) \approx \left({}_k\mathcal{J}^{\gamma}_{a^+}\varphi\right)(x) \tag{6}$$

now when calculating $\left({}_k\mathcal{J}^{\gamma}_{a^+}\widetilde{\varphi}\right)(x)$, we obtain

$$\begin{aligned} _h\left({}_k\mathcal{J}^{\gamma}_{a^+}\varphi\right)_{Re}(x) &= \frac{1}{k\Gamma_k(\gamma)}\int_a^x \frac{\widetilde{\varphi}(t)}{(x-t)^{1-\frac{\gamma}{k}}}\mathrm{d}t \\ &= \frac{\rho^{1-\frac{\gamma}{k}}}{k\Gamma_k(\gamma)}\left\{\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}} \frac{\varphi(x_j)}{(x-t)^{1-\frac{\gamma}{k}}}\mathrm{d}t\right\} \end{aligned} \tag{7}$$

Calculating each integral in (7) we have

$$\int_{x_j}^{x_{j+1}} \frac{\varphi(x_j)}{(x-t)^{1-\frac{\gamma}{k}}}dt = \varphi(x_j)\int_{x_j}^{x_{j+1}} \frac{t^{\rho-1}}{(x-t)^{1-\frac{\gamma}{k}}}dt \tag{8}$$

taking the change of variable

$$u \;=\; x-t \quad , \quad du \;=\; -dt \tag{9}$$

we have

$$\begin{aligned} \varphi(x_j)\int_{x_j}^{x_{j+1}} \frac{1}{(x-t)^{1-\frac{\gamma}{k}}}dt &= -\frac{\varphi(x_j)}{\rho}\int_*^{**}\frac{du}{u^{1-\frac{\gamma}{k}}} \\ &= -\frac{\varphi(x_j)}{\rho}\int_*^{**} u^{\frac{\gamma}{k}-1}du \\ &= -\varphi(x_j)\frac{ku^{\frac{\gamma}{k}}}{\gamma}\Bigg|_*^{**} \\ &= -\varphi(x_j)\frac{k(x-t)^{\frac{\gamma}{k}}}{\gamma}\Bigg|_{x_j}^{x_{j+1}} \\ &= \frac{k}{\gamma}R_{a,j,N}\varphi(x_j) \end{aligned} \tag{10}$$

where

$$R_{a,j,N} = [(a+Nh)-(a+jh)]^{\frac{\gamma}{k}} - [(a+Nh)-(a+(j+1)h)]^{\frac{\gamma}{k}} \tag{11}$$

finally substituting (10) in (3), using (3) and the fact that $\gamma\Gamma_k(\gamma) = \Gamma_K(\gamma + k)$ we have

$$
\begin{aligned}
{}_h\left({}_k\mathcal{J}^{\gamma}_{a^+}\varphi\right)_{Re}(x) &= \frac{1}{k\Gamma_k(\gamma)}\left\{\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}\frac{\varphi(x_j)}{(x-t)^{1-\frac{\gamma}{k}}}\mathrm{d}t\right\} \\
&= \frac{1}{k\Gamma_k(\gamma)}\left\{\sum_{j=0}^{N-1}\frac{k}{\rho\gamma}R_{a,j,N}\varphi(x_j)\right\} \\
&= \frac{1}{\gamma\Gamma_k(\gamma)}\sum_{j=0}^{N-1}R_{a,j,N}\varphi(x_j) \qquad (12)\\
&= \frac{1}{\Gamma_k(\gamma+k)}\sum_{j=0}^{N-1}R_{a,j,N}\varphi(x_j) \\
&= \frac{1}{(k)^{\frac{\gamma}{k}}\Gamma\left(\frac{\gamma}{k}+1\right)}\sum_{j=0}^{N-1}R_{a,j,N}\varphi(x_j)
\end{aligned}
$$

Summarizing the above we have to

Proposition 2.1 *Let $-\infty \leq a < b < \infty$, $\rho, k, \gamma > 0$, $\varphi \in L(a,b)$, $x > a, \gamma \neq -k(kl+1)$ for everything $k \in \mathbb{Z}^+$, $h > 0$, the numerical approximation of $\left({}^{\rho}_{k}\mathcal{J}^{\gamma}_{a^+}\varphi\right)(x)$ with jump h by means of approximations rectangular is given and denoted by*

$$
{}_h\left({}_k\mathcal{J}^{\gamma}_{a^+}\varphi\right)_{Re}(x) = \frac{1}{k^{\frac{\gamma}{k}}\Gamma\left(\frac{\gamma}{k}+1\right)}\sum_{j=0}^{N-1}R_{a,j,N}\varphi(x_j) \qquad (13)
$$

where

$$
R_{a,j,N} = [(a+Nh)-(a+jh)]^{\frac{\gamma}{k}} - [(a+Nh)-(a+(j+1)h)]^{\frac{\gamma}{k}} \qquad (14)
$$

in the particular case that $a = 0$ we have that

$$
{}_h\left({}_k\mathcal{J}^{\gamma}_{0^+}\varphi\right)_{Re}(x) = \frac{h^{\frac{\gamma}{k}}}{k^{\frac{\gamma}{k}}\Gamma\left(\frac{\gamma}{k}+1\right)}\sum_{j=0}^{N-1}R_{0,j,N}\varphi(x_j) \qquad (15)
$$

where

$$
R_{j,N} = [N-j]^{\frac{\gamma}{k}} - [N-(j+1)]^{\frac{\gamma}{k}} \qquad (16)
$$

in the case that $k = 1$ we have the numerical approximation of the generalized fractional integral $\mathcal{J}^{\gamma}_{a^+}$ is particular case in (KATUGAMPOLA, 2014) given by:

$$
{}_h\left(\mathcal{J}^{\gamma}_{a^+}\varphi\right)_{Re}(x) = \frac{1}{\Gamma(\gamma+1)}\sum_{j=0}^{N-1}Rk_{a,j,N}\varphi(x_j) \qquad (17)
$$

where

$$
Rk_{a,j,N} = [(a+Nh)-(a+jh)]^{\gamma} - [(a+Nh)-(a+(j+1)h)]^{\gamma} \qquad (18)
$$

and when $\rho = 1$ we obtain the numerical approximation of the fractional integral k-Riemann-Liouville ${}_k\mathcal{J}^{\gamma}_{a^+}$ given in (MUBEEN; HABIBULLAH, 2012) by Mubeen and Habibullah.

$$ {}_h\left({}_k\mathcal{J}^{\gamma}_{a^+}\varphi\right)_{Re}(x) = \frac{h^{\frac{\gamma}{k}}}{(k)^{\frac{\gamma}{k}}\Gamma\left(\frac{\gamma}{k}+1\right)}\sum_{j=0}^{N-1} Rkrl_{a,j,N}\varphi(x_j) \tag{19} $$

where

$$ Rkrl_{a,j,N} = [N-j]^{\frac{\gamma}{k}} - [N-j-1]^{\frac{\gamma}{k}} \tag{20} $$

finally if $k = 1$ we obtain the numerical approximation of the integral of classical Riemann-Liouville $\mathcal{J}^{\gamma}_{a^+}$ given by

$$ {}_h\left(\mathcal{J}^{\gamma}_{a^+}\varphi\right)_{Re}(x) = \frac{h^{\gamma}}{\Gamma(\gamma+1)}\sum_{j=0}^{N-1} Rrl_{a,j,N}\varphi(x_j) \tag{21} $$

where

$$ Rkrl_{a,j,N} = [N-j]^{\gamma} - [N-j-1]^{\gamma} \tag{22} $$

to illustrate all of the above we will compare some integrals with their respective approximations numerical, for this let's see some examples of fractional integrals, the following example is weak version of Theorem 2.4 in (SARIKAYA et al., 2016).

Example 2.2 *Let γ, β, k. Then we have*

$$ \left({}_k\mathcal{J}^{\gamma}_{0^+}(2t^{\frac{\beta}{k}-1})\right)(x) = \frac{2x^{\frac{\gamma+\beta}{k}-1}\Gamma\left(\frac{\beta}{k}\right)}{k^{\frac{\gamma}{k}}\Gamma\left(\frac{\gamma+\beta}{k}\right)} \tag{23} $$

Solution:
By definition and linearity of the fractional integral, we calculate yet

$$ \begin{aligned} \left({}_k\mathcal{J}^{\gamma}_{0^+}(t^{\frac{\beta}{k}-1})\right)(x) &= \frac{1}{k\Gamma_k(\gamma)}\int_0^x \frac{t^{\rho-1}}{(x-t)^{1-\frac{\gamma}{k}}}t^{\frac{\beta}{k}-1}dt \\ &= \frac{1}{k\Gamma_k(\gamma)}\int_0^x \frac{t^{\frac{\beta}{k}-1}}{x^{1-\frac{\gamma}{k}}(1-\frac{1}{x})^{1-\frac{\gamma}{k}}}dt \end{aligned} \tag{24} $$

using variable change

$$ y \;=\; 1-\tfrac{t}{x} \quad , \quad dy \;= -\tfrac{1}{x}dt \tag{25} $$

we have

$$ \begin{aligned} \left({}_k\mathcal{J}^{\gamma}_{0^+}(t^{\frac{\beta}{k}-1})\right)(x) &= \frac{(x)^{\frac{(\gamma+\beta)}{k}-1}}{k\Gamma_k(\gamma)}\int_0^1 y^{\frac{\beta}{k}-1}(1-y)^{\frac{\gamma}{k}-1}dy \\ &= \frac{(x^{\rho})^{\frac{\gamma+\beta}{k}-1}}{k\Gamma_k(\gamma)}B\left(\frac{\beta}{k},\frac{\gamma}{k}\right) \\ &= \frac{x^{\frac{\gamma+\beta}{k}-1}\Gamma\left(\frac{\beta}{k}\right)\Gamma\left(\frac{\gamma}{k}\right)}{k^{\frac{\gamma}{k}}\Gamma\left(\frac{\gamma}{k}\right)} \\ &= \frac{x^{\frac{\gamma+\beta}{k}-1}\Gamma\left(\frac{\beta}{k}\right)}{k^{\frac{\gamma}{k}}\Gamma\left(\frac{\beta+\gamma}{k}\right)} \end{aligned} \tag{26} $$

where $B_k(\gamma, \beta)$ is the function k-beta with $B_k(\gamma, \beta) = \frac{\Gamma_k(\gamma)\Gamma_k(\beta)}{\Gamma_k(\gamma+\beta)}$ In particular taking for example

$$k = 0.5, \gamma = 0.5, a = 0, \beta = 0.6 \tag{27}$$

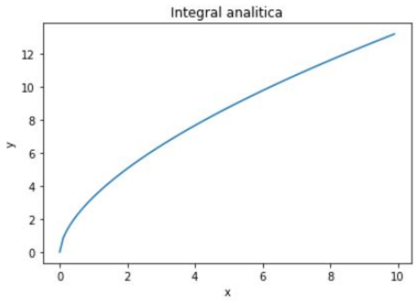
you have

$$\varphi(x) = x^{\frac{1}{10}} \tag{28}$$

then from the above

$$\left({}_{\frac{1}{2}}\mathcal{J}_{0^+}^{\frac{1}{2}}(2t^{\frac{1}{10}})\right)(x) = \frac{4x^{\frac{3}{5}}\Gamma\left(\frac{6}{5}\right)}{\Gamma\left(\frac{11}{5}\right)} \tag{29}$$

graphing this integral in $[0, 10]$ *we have*

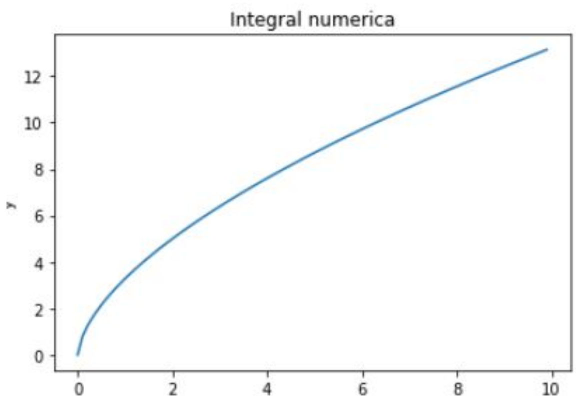


■

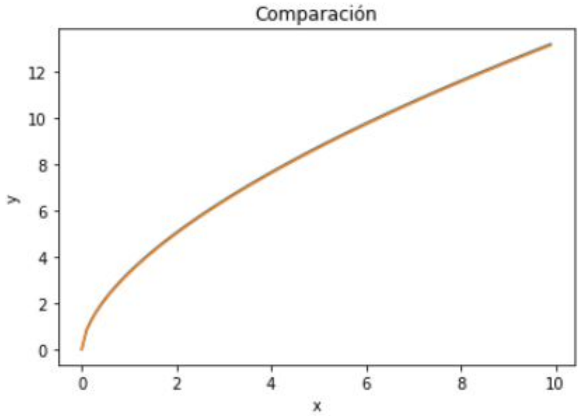
Now let's see the numerical approximation of the integral in the previous example, for this we performed Python code in which the function $hJRe$ was programmed who is in charge of carrying out the numerical approximation, which has for inputs,$h, k, \gamma, a, \varphi, x$, for convenience in programming the variables were renamed as follows

$$h = h, k = k, \gamma = g, a = a, \beta = b, \varphi = fi, x = x$$

and shows as a result the approximation for $h = 0.001$



and graphing the two on the same plane



they become indistinguishable As already mentioned in proposition 2.1, the previous code in addition to giving an approximation numerical of the integral k-Riemann-Liouville fractional integral.

3 Generalized trapezoid rule for numerical approximation of the integral operator

If we continue with the usual integration theory we should now consider a function $\widetilde{\widetilde{\varphi}}$ such that in each subinterval $[x_j, x_{j+1}]$ we should have the interpolation polynomial of degree 1 however, given the fractional nature of the operator ${}_k\mathcal{J}^{\gamma}_{a^+}$ we find ourselves in need of generalize this trapezoidal method and consider functions of potential form with power ρ, that is, functions of the form $c_1 + c_2 t$, for this we will first find a function of this form that go through the points $[x_j, x_{j+1}]$ making an analogy with the point slope form of the line is more 119 It is convenient to write it as 1 + 2 (-), this leads us to the following system of equations

$$\begin{aligned} \varphi(x_j) &= c_1 + c_2(x_j - x_j) \\ \varphi(x_{j+1}) &= c_1 + c_2(x_{j+1} - x_j) \end{aligned} \tag{30}$$

With which

$$\begin{aligned} c_1 &= \varphi(x_j) \\ c_2 &= \frac{\varphi(x_{j+1}) - \varphi(x_j)}{x_{j+1} - x_j} \end{aligned} \tag{31}$$

thus we obtain a potential piece wise approximation φ given by:

$$\widetilde{\widetilde{\varphi}}(t) = \varphi(x_j) + \frac{\varphi(x_{j+1}) - \varphi(x_j)}{x_{j+1} - x_j^{\rho}}(t - x_j), \qquad x_j \leq t \leq x_{j+1} \tag{32}$$

by $j = 0, \ldots, N-1$.
Now we numerically approximate $\left({}_k\mathcal{J}^{\gamma}_{a^+}\varphi\right)(x)$ through $\left({}_k\mathcal{J}^{\gamma}_{a^+}\widetilde{\widetilde{\varphi}}\right)(x)$ which we will denote:

$${}_h\left({}_k\mathcal{J}^{\gamma}_{a^+}\varphi\right)_{Tr}(x) = \left({}_k\mathcal{J}^{\gamma}_{a^+}\widetilde{\widetilde{\varphi}}\right)(x) \approx \left({}_k\mathcal{J}^{\gamma}_{a^+}\varphi\right)(x) \tag{33}$$

To calculate $\left({}_k\mathcal{J}_{a^+}^{\gamma}\widetilde{\widetilde{\varphi}}\right)(x)$, we have

$$\begin{aligned} {}_h\left({}_k\mathcal{J}_{a^+}^{\gamma}\varphi\right)_{Tr}(x) &= \frac{1}{k\Gamma_k(\gamma)}\int_a^x \frac{\widetilde{\widetilde{\varphi}}(t)}{(x-t)^{1-\frac{\gamma}{k}}}\mathrm{d}t \\ &= \frac{1}{k\Gamma_k(\gamma)}\left\{\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}} \frac{t^{\rho-1}}{(x-t)^{1-\frac{\gamma}{k}}}\left[\varphi(x_j)+\frac{\varphi(x_{j+1})-\varphi(x_j)}{x_{j+1}-x_j}(t-x_j)\right]\mathrm{d}t\right\} \end{aligned} \tag{34}$$

to calculate the integral in (34) we have

$$\begin{aligned} &\int_{x_j}^{x_{j+1}} \frac{t^{\rho-1}}{(x-t)^{1-\frac{\gamma}{k}}}\left[\varphi(x_j)+\frac{\varphi(x_{j+1})-\varphi(x_j)}{x_{j+1}-x_j}(t-x_j)\right]dt \\ &= \varphi(x_j)\int_{x_j}^{x_{j+1}} \frac{1}{(x-t)^{1-\frac{\gamma}{k}}}dt + \frac{\varphi(x_{j+1})-\varphi(x_j)}{x_{j+1}-x_j}\int_{x_j}^{x_{j+1}} \frac{1}{(x-t)^{1-\frac{\gamma}{k}}}(t-x_j)dt \\ &= \varphi(x_j)\mathbf{I} + \frac{\varphi(x_{j+1})-\varphi(x_j)}{x_{j+1}-x_j}\mathbf{II} \end{aligned} \tag{35}$$

by I using variable change

$$u = x-t \quad , \quad du = -dt \tag{36}$$

we have

$$\begin{aligned} I &= \int_{x_j}^{x_{j+1}} \frac{1}{(x-t)^{1-\frac{\gamma}{k}}}dt \\ &= -\int_*^{**} \frac{du}{u^{1-\frac{\gamma}{k}}} \\ &= -\int_*^{**} u^{\frac{\gamma}{k}-1}du \\ &= -\frac{ku^{\frac{\gamma}{k}}}{\gamma}\Bigg|_*^{**} \\ &= -\frac{k(x-t)^{\frac{\gamma}{k}}}{\rho\gamma}\Bigg|_{x_j}^{x_{j+1}} \\ &= \frac{k}{\gamma}\left[(x-x_{j+1})^{\frac{\gamma}{k}} - (x-x_j)^{\frac{\gamma}{k}}\right] \end{aligned} \tag{37}$$

by II we integrate by parts with

$$\begin{aligned} u &= t-x_j \quad , \quad dv = \frac{1}{(x-t)^{1-\frac{\gamma}{k}}}dt \\ du &= dt \quad , \quad v = -\frac{k(x-t)^{\frac{\gamma}{k}}}{\gamma} \end{aligned} \tag{38}$$

so

$$
\begin{aligned}
II &= -\frac{k(x-t)^{\frac{\gamma}{k}}(t-x_j)}{\gamma}\Bigg|_{x_j}^{x_{j+1}} + \frac{k}{\gamma}\int_{x_j}^{x_{j+1}}(x-t)^{\frac{\gamma}{k}}dt \\
&= -\frac{k(x-x_{j+1})^{\frac{\gamma}{k}}(x_{j+1}-x_j)}{\gamma} + \frac{k}{\gamma}\int_{*}^{**} w^{\frac{\gamma}{k}}dw \\
&= \frac{k}{\gamma}\left\{-(x-x_{j+1})^{\frac{\gamma}{k}}(x_{j+1}-x_j) + \left[\frac{w^{\frac{\gamma}{k}+1}}{\frac{\gamma}{k}+1}\right]_{*}^{**}\right\} \\
&= \frac{k}{\gamma}\left\{-(x-x_{j+1})^{\frac{\gamma}{k}}(x_{j+1}-x_j) + \left[\frac{(x-t)^{\frac{\gamma}{k}+1}}{\frac{\gamma}{k}+1}\right]_{x_j}^{x_{j+1}}\right\} \\
&= \frac{k}{\gamma}\left\{-(x-x_{j+1})^{\frac{\gamma}{k}}(x_{j+1}-x_j) + \frac{k}{\gamma+k}\left[(x-(x_{j+1})^{\frac{\gamma}{k}+1} - (x-x_j)^{\frac{\gamma}{k}+1}\right]\right\} \\
&= \frac{k^2\left[(x-x_{j+1})^{\frac{\gamma}{k}+1} - (x-x_j)^{\frac{\gamma}{k}+1}\right]}{\gamma(\gamma+k)} - \frac{k(x-x_{j+1})^{\frac{\gamma}{k}}(x_{j+1}-x_j)}{\gamma}
\end{aligned}
\tag{39}
$$

where was it used

$$
w \;=\; x-t \quad , \quad dw \;=\; -dt \tag{40}
$$

substituting (39) and (37) in (35) we have

$$
\begin{aligned}
&\int_{x_j}^{x_{j+1}} \frac{1}{(x-t)^{1-\frac{\gamma}{k}}}\left[\varphi(x_j) + \frac{\varphi(x_{j+1})-\varphi(x_j)}{x_{j+1}-x_j}(t-x_j)\right]dt \\
&= \varphi(x_{j+1})\left\{\frac{k^2\left[(x-x_{j+1})^{\frac{\gamma}{k}+1} - (x-x_j)^{\frac{\gamma}{k}+1}\right]}{\gamma(\gamma+k)(x_{j+1}-x_j)} - \frac{k(x-x_{j+1})^{\frac{\gamma}{k}}}{\gamma}\right\} \\
&\quad - \varphi(x_j)\left\{\frac{k^2\left[(x-(x_{j+1}))^{\frac{\gamma}{k}+1} - (x-x_j)^{\frac{\gamma}{k}+1}\right]}{\rho\gamma(\gamma+k)(x_{j+1}-x_j)} + \frac{k(x-x_j)^{\frac{\gamma}{k}}}{\gamma}\right\}
\end{aligned}
\tag{41}
$$

Finally adding for $0 \le j \le N-1$ it follows that

$$
\begin{aligned}
&= \varphi(x_j)\frac{k^2}{\gamma(\gamma+k)}\sum_{j=1}^{N-1} Tr_{a,j,N} \\
&\quad - \varphi(x_N)\left\{\frac{k^2(x_N-x_{N-1})^{\frac{\gamma}{k}}}{\gamma(\gamma+k)}\right\} \\
&\quad - \varphi(x_0)\left\{\frac{k^2\left[(x_N-x_1)^{\frac{\gamma}{k}+1} - (x_N-x_0)^{\frac{\gamma}{k}+1}\right]}{\gamma(\gamma+k)(x_1-x_0)} + \frac{k(x_N-x_0)^{\frac{\gamma}{k}}}{\gamma}\right\}
\end{aligned}
\tag{42}
$$

where

$$
Tr_{a,j,N} = \frac{(x-x_j)^{\frac{\gamma}{k}+1} - (x-x_{j-1})^{\frac{\gamma}{k}+1}}{(x_j-x_{j-1})} - \frac{(x-x_{j+1})^{\frac{\gamma}{k}+1} - (x-x_j)^{\frac{\gamma}{k}+1}}{(x_{j+1}-x_j)} \tag{43}
$$

by $N \geq 2$, in the case that $N = 0$ in (34) $x = a$ and so on ${}_h\left({}_k\mathcal{J}^{\gamma}_{a^+}\varphi\right)_{Tr}(a) = 0$ and in the case that $N = 1$ we have to

$$\begin{aligned}
{}_h\left({}_k\mathcal{J}^{\gamma}_{a^+}\varphi\right)_{Tr}(x) &= \frac{1}{k\Gamma_k(\gamma)}\left\{\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}\frac{1}{(x-t)^{1-\frac{\gamma}{k}}}\left[\varphi(x_j)+\frac{\varphi(x_{j+1})-\varphi(x_j)}{x_{j+1}-x_j}(t-x_j)\right]\mathrm{d}t\right\}\\
&= \frac{1}{k\Gamma_k(\gamma)}\left(\varphi(x_0)-\varphi(x_1)\right)\left\{\frac{k^2(x_1^{\rho}-(x_0))^{\frac{\gamma}{k}}}{\gamma(\gamma+k)}\right\}-\varphi(x_0)\left\{\frac{k(x_1-x_0)^{\frac{\gamma}{k}}}{\gamma}\right\} \qquad (44)\\
&= \frac{1}{k\Gamma_k(\gamma)}\frac{k(x_1-x_0)^{\frac{\gamma}{k}}}{\gamma}\left\{\frac{k\left(\varphi(x_0)-\varphi(x_1)\right)}{(\gamma+k)}-\varphi(x_0)\right\}
\end{aligned}$$

finally joining the above we have a numerical approximation of the integral operatorl

Proposition 3.1 *Let* $-\infty \leq a < b < \infty$, $\rho, k, \gamma > 0$, $\varphi \in L(a,b)$, $x > a, \gamma \neq -k(kl+1)$ *for everything* $k \in \mathbb{Z}^+$, $h > 0$, *the numerical approximation of* $\left({}^{\rho}_{k}\mathcal{J}^{\gamma}_{a^+}\varphi\right)(x)$ *with jump* h *by means of approximations generalized trapezoidal is given and denoted by*

$$\begin{aligned}
{}_h\left({}_k\mathcal{J}^{\gamma}_{a^+}\varphi\right)_{Tr}(x) = C_{k,\gamma}\varphi(x_j)\sum_{j=1}^{N-1}Tr_{a,j,N}&\\
- C_{k,\gamma}\varphi(x_N)(x_N-(x_{N-1}))^{\frac{\gamma}{k}}& \qquad (45)\\
- C_{k,\gamma}\varphi(x_0)\left\{\frac{\left[(x_N-x_1)^{\frac{\gamma}{k}+1}-(x_N-x_0)^{\frac{\gamma}{k}+1}\right]}{(x_1-x_0)}+\frac{(\gamma+k)(x_N-x_0)^{\frac{\gamma}{k}}}{k}\right\}&
\end{aligned}$$

where

$$Tr_{a,j,N} = \frac{x-(x_j)^{\frac{\gamma}{k}+1}-(x-x_{j-1})^{\frac{\gamma}{k}+1}}{(x_j-x_{j-1})}-\frac{(x-x_{j+1})^{\frac{\gamma}{k}+1}-(x-x_j)^{\frac{\gamma}{k}+1}}{(x_{j+1}-x_j)} \qquad (46)$$

and

$$C_{k,\gamma} = \frac{k}{\gamma(\gamma+k)\Gamma_k(\gamma)} \qquad (47)$$

by $N \geq 2$, *if* $N = 0$ ${}_h\left({}_k\mathcal{J}^{\gamma}_{a^+}\varphi\right)_{Tr}(x) = 0$ *y* $N = 1$

$${}_h\left({}_k\mathcal{J}^{\gamma}_{a^+}\varphi\right)_{Tr}(x) = C_{k,\gamma}(x_1-x_0)^{\frac{\gamma}{k}}\left\{(\varphi(x_0)-\varphi(x_1))-\frac{(\gamma+k)\varphi(x_0)}{k}\right\}. \qquad (48)$$

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