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A study of forced oscillations via Hilfer fractional derivative

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Abstract

The present study seeks to understand the forced oscillations through modeling via fractional differential equation, using the derivative according to Hilfer and representing the external force as a succession of delta Dirac functions. This formulation allows recovering the solutions according to Caputo and Riemann-Liouville. The results obtained show that both Caputo and Riemann-Liouville solutions coincide when we recover the entire order of the derivative. Also, by switching the order of the derivative it is possible to simulate damping.

Keywords: Fractional Calculus. Forced oscillator. Hilfer fractional derivative. Dirac delta function.

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1 Introduction

When a body of mass m performs a periodic movement around a fixed point this movement is said to be oscillatory. A special type of oscillatory movement is the harmonic movement. The movements are placed in this classification oscillators in which the acceleration and the resulting force acting on the body are proportional and opposite to its displacement (simple harmonic movement, SHM, in short). Separate the study in other cases: the oscillations with damping (when harmonic movement occurs in the presence of friction window) and forced oscillations (when the system oscillates under the action of a periodic external force), or rather, “oscillator receives energy from the external force at which it is subjected and dissipates an amount of energy due to the existence of the friction force"(DINIZ, 2019). Unlike the present study, in (NUSSENZVEIG, 2013, p.120) the $x(t)$ function in the SHM is determined from the analysis of the energy conservation associated with the spring-mass system, since “a total energy E remains constant, oscillating between the kinetic form and the potential form" (NUSSENZVEIG, 2013, p.121).

To understand this dynamics it’s necessary to equate the dimensions acting on the particle and determine the time function of the displacement from its initial conditions of position and velocity. In general, this process is carried out by solving an ordinary differential equation. In other words, applying Newton’s second law to the description of the movement of a body, it is possible to obtain a time-dependent analytical expression for such quantities (displacement, natural frequency of the oscillator, period of oscillation, etc.).

Furthermore, this modeling is done using integer-order derivative operators, that is, $\frac{\mathrm{d}}{\mathrm{d}t}$ and $\frac{\mathrm{d}^2}{\mathrm{d}t^2}$. With regard to the harmonic oscillator, the modeling occurs through a second order ordinary differential equation with constant coefficients, according to the following equation:

$$m\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}+\beta\frac{\mathrm{d}x(t)}{\mathrm{d}t}+kx(t)=0, \tag{1}$$

where m represents the mass in kg, β is the damping coefficient in kg/s, k the spring constant in N/m and x is the displacement of the oscillator relative to its position of balance.

Fractional calculus (FC) makes it possible to “replace integer-order derivatives by non-integer order derivatives, generally with order less than or equal to the order of the original derivatives, so that the usual solution can be retrieved as a particular case" (THEODORO; CAMARGO, 2020). It is important to highlight that “the fractional operator reflects intrinsic dissipative processes that are complicated enough by nature" (STANISLAVSKY, 2006), but this relationship is not yet fully specified.

While in integer order calculus (sometimes called classical calculus) there is a consensus on the definition of derivative and integral operators, in FC there is no single formulation of these operators. Also, each formulation can be better suited to a specific physical context. There are several formulations for the fractional derivative, some few of them are Riemann-Liouville, Caputo, Grïwald-Letnikov, Weyl and Hilfer.

Therefore, the study of oscillators in FC is performed by solving a fractional differential equation, that is, differential equation whose derivative operator can takes a non-integer order. Following this line of research, this work aims to study the forced oscillator by a periodic external force via Hilfer fractional derivative. We also study the curves that describe the solutions found when the order of the derivative is changed.

Regarding the methodology, the inverse Laplace transform will be used to find a solution of the fractional differential equation for the forced oscillator. However, first it is essential to present the formulation and demonstrate some results. More precisely, we will first present the Mittag-Leffler function that plays an important role in FC, similar to the exponential function in classical calculus. Then, the Riemann-Liouville fractional integral will be defined, as well as the Riemann-Liouville fractional derivative, Caputo and, finally, Hilfer formulation (which recovers both). At the end of the work, the inverse Laplace transform methodology will be used to find the solution of the fractional differential equation describing the fractional forced oscillator.

2 Mittag-Leffler function

Just as in classical calculus a solution of many ordinary differential equation is expressed in terms of exponential function, in FC the Mittag-Leffler function plays the same role. In addition, to generalizing the exponential function, this function have interesting properties related to the fractional derivative operator. It is worth noting that several generalizations of Mittag-Leffler function have been proposed. In this work, we use only the Mittag-Leffler functions of one and two parameters. Another importante function is the three-parameter Mittag-Leffler function as proposed by Prabhakar in 1971 (PRABHAKAR, 1971).

Definition 2.1 *Let $E_{\alpha,\beta}(z)$ be a complex function with two complex parameters α and β, with $Re(\alpha) > 0$ and $\beta \in \mathbb{C}$ such that*

$$E_{\alpha,\beta}(z) = \sum_{n=1}^{\infty} \frac{z^n}{\Gamma(\alpha z + \beta)}. \tag{2}$$

with $E_{\alpha,\beta}(\cdot)$ a Mittag-Leffler function with two parameters and Γ is the gamma function, while for $\beta = 1$ we obtain the classical Mittag-Leffer function with one parameter, denoted by $E_\alpha(\cdot)$.

The condition $Re(\alpha) > 0$ ensures that the series $E_{\alpha,\beta}(\cdot)$ converges (OLIVEIRA, 2019). For $Re(\alpha) < 0$ the series diverges and for $Re(\alpha) = 0$, $|z| < 1$ we have the well-known geometric series, that is, $E_0(z) = \frac{1}{1-z}$.

Some particular cases stand out:

$$\begin{aligned} E_{1,1}(z) &= \sum_{n=1}^{\infty} \frac{z^n}{\Gamma(n+1)} = \sum_{n=1}^{\infty} \frac{z^n}{n!} = e^z \\ E_{2,1}(-z^2) &= \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n}}{\Gamma(2n+1)} = \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = \cos(z) \\ E_{2,2}(-z^2) &= \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n}}{\Gamma(2n+2)} = \frac{\sum_{n=1}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}}{z} = \frac{\sin(z)}{z}. \end{aligned} \tag{3}$$

Theorem 2.1 *Let $t \in \mathbb{R}$ and a a real constant. We have*

$$t^{\beta-1} E_{\alpha,\beta}\{\pm a t^\alpha\} \div \frac{1}{s^\beta} \frac{s^\alpha}{(s^\alpha \mp a)}, \tag{4}$$

for $\alpha > 0$ and $\beta > 0$. The symbol $\div$ in this context means that the first member is the inverse Laplace transform of the second member.

Proof. To prove the result, we first take the Laplace transform, obtaining

$$\mathcal{L}[t^{\beta-1}E_{\alpha,\beta}(\pm at^{\alpha})] = \int_0^{\infty} e^{-st}t^{\beta-1}\sum_0^{\infty}\frac{(\pm at^{\alpha})^n}{\Gamma(\alpha n+\beta)}\mathrm{d}t = \sum_{n=0}^{\infty}\frac{(\pm a)^n}{\Gamma(\alpha n+\beta)}\int_0^{\infty} t^{\alpha n+\beta-1}e^{-st}\mathrm{d}t.$$

Introducing a change $y = st$, implies that:

$$= \sum_{n=0}^{\infty}\frac{1}{s^{\beta}}\frac{(\pm a)^n}{\Gamma(\alpha n+\beta)}\frac{1}{s^{\alpha n}}\int_0^{\infty} e^{-y}y^{\alpha n+\beta-1}\mathrm{d}y,$$

with the integral appearing in the second member being a gamma function. So, we have

$$\mathcal{L}[t^{\beta-1}E_{\alpha,\beta}(\pm at^{\alpha})] = \frac{1}{s^{\beta}}\sum_{n=0}^{\infty}\frac{(\pm a)^n}{\Gamma(\alpha n+\beta)\,s^{\alpha n}}\Gamma(\alpha n+\beta) = \frac{1}{s^{\beta}}\sum_{n=0}^{\infty}\left(\pm\frac{a}{s^{\alpha}}\right)^n.$$

Besides, for $\left|\frac{a}{s^{\alpha}}\right| < 1$ a convergent geometric series is obtained. Therefore,

$$\mathcal{L}[t^{\beta-1}E_{\alpha,\beta}(\pm at^{\alpha})] = \frac{1}{s^{\beta}}\frac{1}{1\mp\frac{a}{s^{\alpha}}} = \frac{1}{s^{\beta}}\frac{s^{\alpha}}{(s^{\alpha}\mp a)},$$

which concludes the result.

3 Fractional derivatives

In this section, before we present fractional derivatives, we discuss the Riemann-Liouville fractional integral. Only three types of fractional derivatives are presented.

3.1 Riemann-Liouville fractional integral

In order to define the formulations of fractional derivatives, the Riemann Liouville fractional integral must first be well defined.

Definition 3.1 *The Riemann-Liouville fractional integral of order α of a function f is given by,*

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t f(\tau)\,(t-\tau)^{\alpha-1}\,d\tau, \tag{5}$$

with $Re(\alpha) > 0$.

This integral can also be understood as a Laplace convolution product of the f function with a Gel'fand-Shilov function (TEODORO; OLIVEIRA, 2017).

3.2 Riemann-Liouville fractional derivative

Definition 3.2 *Let α be a complex number such that $Re(\alpha)$ and n the smallest integer greater than $Re(\alpha)$, thus $n-1 < Re(\alpha) \leq n$. The Riemann-Liouville fractional derivative of a sufficiently well-behaved function f is given by,*

$$D^{\alpha}f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}I^{n-\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}}\mathrm{d}\tau. \tag{6}$$

3.3 Caputo fracional derivative

Definition 3.3 *Let T, $\alpha \in \mathbb{R}^+$ and $n = \min\{k \in \mathbb{N}/k \geq \alpha\}$. Caputo fractional derivative on the left, of a function f is defined as*

$$
{}^{C}D^{\alpha}_{0^+}f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-1-\alpha} f^{(n)}(\tau)\mathrm{d}\tau, \; n-1 < \alpha < n \\ f^{(n)}(t), \qquad\qquad\qquad\qquad\qquad\qquad\qquad , n = \alpha. \end{cases} \tag{7}
$$

It is important to note that, differently to the Riemann-Liouville fractional derivative, the Caputo fractional derivative of a constant K is ${}^{C}D^{\alpha}_{0^+}K = 0$.

3.4 Hilfer fractional derivative

Historically, the formulation of the derivative according to Hilfer, or just Hilfer fractional derivative, can be considered recent. This derivative recovers, for particular values of parameters, Riemann-Liouville and Caputo fractional derivatives, and the so-called Weyl derivative, also.

Definition 3.4 *Let $n-1 < \gamma < n$, with $n \in \mathbb{N}$ and $\Lambda = [a, b]$ a closed interval $-\infty \leq a < b \leq +\infty$, $t \in \Lambda$ and $x \in C^m(\Lambda, \mathbb{R})$. The Hilfer fractional operator, denoted by ${}^{H}D^{\gamma,\mu}_{a^+}$ of order γ and type μ with $0 \leq \mu \leq 1$ is given by*

$$
{}^{H}D^{\gamma,\mu}_{a^+} = I^{\mu(n-\gamma)}_{a^+} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n I^{(1-\mu)(n-\gamma)}_{a^+} x(t), \tag{8}
$$

where $I^{\mu}_{a^+}$ is the Riemann-Liouville fractional integral.

Note that for $\mu = 0$ the derivative according to Riemann-Liouville is retrieved while for $\mu = 1$ the derivative according to Caputo is retrieved.

Theorem 3.1 *The Laplace transform of a Hilfer fractional derivative of order $0 < \alpha \leq 1$ and type $0 \leq \mu \leq 1$ is given by*

$$
\mathcal{L}\left[{}^{H}D^{\alpha,\mu}_{0+}f(t)\right] = s^{\alpha}F(s) - s^{\mu(\alpha-1)}I^{(1-\mu)(1-\alpha)}_{0+}f(0^+), \tag{9}
$$

where $I^{(1-\mu)(1-\alpha)}_{0+}f(0^+) = \lim_{t\to 0^+} I^{(1-\mu)(1-\alpha)}_{0+}f(t)$. On the other hand, for the order $1 < \gamma \leq 2$ the result is

$$
\mathcal{L}\left[{}^{H}D^{\gamma,\mu}_{0+}x(t)\right] = s^{\gamma}F(s) - s^{\mu(\gamma-2)}I^{(1-\mu)(2-\gamma)-1}_{0+}f(t)|_{t=0} - s^{1+\mu(\gamma-2)}I^{(1-\mu)(2-\gamma)}_{0+}f(t)|_{t=0}. \tag{10}
$$

Proof. Taking the Laplace transform on both sides, we get

$$
\mathcal{L}\left[{}^{H}D^{\alpha,\mu}_{0+}f(t)\right] = \mathcal{L}\left[I^{\mu(1-\alpha)}_{0+}\frac{\mathrm{d}}{\mathrm{d}t}I^{(1-\mu)(1-\alpha)}_{0+}f(t)\right], \tag{11}
$$

with $A(t) = \frac{\mathrm{d}}{\mathrm{d}t}I^{(1-\mu)(1-\alpha)}_{0+}f(t)$. Remembering the Laplace transform of an integral, we have

$$
\mathcal{L}\left[I^{\mu(1-\alpha)}_{0+}(A(t))\right] = \frac{\mathcal{L}[A(t)]}{s^{\mu(1-\alpha)}}, \tag{12}
$$

and for the derivative, we obtain

$$\mathcal{L}\left[A(t)\right] = \mathcal{L}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(I_{0+}^{(1-\mu)(1-\alpha)}f(t)\right)\right] = s\mathcal{L}\left[I_{0+}^{(1-\mu)(1-\alpha)}f(t)\right] - I_{0+}^{(1-\mu)(1-\alpha)}f(0^+). \quad (13)$$

Also, again using the Laplace transform of an integral, one can write

$$\mathcal{L}\left[I_{0+}^{(1-\mu)(1-\alpha)}f(t)\right] = \frac{F(s)}{s^{(1-\mu)(1-\alpha)}}. \quad (14)$$

Replacing the expressions (13) and (14) in the equation (12), implies

$$\mathcal{L}\left[{}^{H}D_{0+}^{\alpha,\mu}f(t)\right] = s^{\alpha}F(s) - s^{\mu(\alpha-1)}I_{0+}^{(1-\mu)(1-\alpha)}f(0^+). \quad (15)$$

The reasoning above permits to find the expression for the Laplace transform in the case $1 < \gamma \leq 2$ and the same type $0 < \mu < 1$. So, we have (TOMOVSKI; HILFER; SRIVASTAVA, 2010)

$$\mathcal{L}\left[{}^{H}D_{0^+}^{\gamma,\mu}x(t)\right] = s^{\gamma}F(s) - s^{\mu(\gamma-2)}I_{0^+}^{(1-\mu)(2-\gamma)-1}f(t)|_{t=0} - s^{1+\mu(\gamma-2)}I_{0^+}^{(1-\mu)(2-\gamma)}f(t)|_{t=0}. \quad (16)$$

4 Forced oscillations

Traditionally to illustrate the simple harmonic oscillator, a spring-mass system, that moves freely with the kinetic energy being conserved during the movement is used. Eq.(1) occurs when $\beta = 0$, that is, when there is no damping term.

Although, a big issue that arises when dealing with arbitrary order derivative operators is the interpretation of the units of measure referring to the operators. This can be verified in (GÓMEZ-AGUILAR et al., 2012) where an arbitrary parameter σ of temporal dimension (second unit of measure) is introduced to ensure that all quantities have their dimensions adjusted according to the order γ allowing, thus, study the behavior of solutions for intermediate values of γ. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t} \rightarrow \frac{1}{\sigma^{1-\gamma}}\frac{\mathrm{d}^{\gamma}}{\mathrm{d}t^{\gamma}},$$

still in (GÓMEZ-AGUILAR et al., 2012) this derivative operator refers to Caputo's formulation. The same strategy will be used here, but for Hilfer's formulation. The Eq.(1) can be written as

$$\frac{m}{\sigma^{2-\gamma}}{}^{H}D_{0^+}^{\gamma,\mu}\left(x\left(t\right)\right) + \frac{\beta}{\sigma^{1-\alpha}}{}^{H}D_{0^+}^{\alpha,\mu}\left(x\left(t\right)\right) + kx(t) = 0, \quad (17)$$

where $0 < \alpha \leq 1$, $1 < \gamma \leq 2$ and $0 \leq \mu \leq 1$.

However, the second member zero indicates that there are no external forces adding energy to the system. For forced oscillations, a periodic function must be added in second member of Eq.(1).

It is important to cite interesting fractional forced oscillator approaches, such as (ŁABĘDZKI; PAWLIKOWSKI; RADOWICZ, 2019). In this work the authors use a function $f(t) = A\sin(\omega_0 t)$ describing oscillating excitation (external charge), where A is the amplitude and ω_0 is the external load frequency. Moreover, in this work they use the Riemann-Liouville fractional derivative in order to understand vibrations in continuous structures of viscoelastic materials. A similar approach is found in (CHUNG; JUNG, 2014), where the driving force is expressed in terms of a Mittag-Leffler cosine function, but using the Caputo fractional derivative.

In (PAROVIK, 2020) a corresponding numerical study is carried out using the Caputo fractional derivative. From the finite difference method to solve equations and other numerical methods to build oscillograms and phase trajectories in order to understand these solutions, the study provides further evidence that the order of the derivative is responsible for the intensity of energy dissipation in fractional vibrational systems.

Thus, in this study the external force will also be modeled as a periodic function as in (ŁABĘDZKI; PAWLIKOWSKI; RADOWICZ, 2019) and (PAROVIK, 2020). For this, the idea presented in the (DUTRA; RIBEIRO; PORTO, 2018) will be used. More precisely, the situation is illustrated as follows: a child being periodically propelled on a swing. To describe the external force that provokes this succession of impulses of same magnitude it is important to comment on a particular distribution, the so-called Dirac comb. Let J_0 be the strength of each given impulse, so the strength of the force $F(t)$ is represented by the following distribution

$$F(t) = J_0 \sum_{n=1}^{N} \delta\left(t - n\tau\right), \tag{18}$$

where τ is the time interval of each push, N the number of pushes and δ the Dirac delta function.

In this way, the fractional differential equation to be solved can be expressed as:

$$\frac{m}{\sigma^{2-\gamma}}\, {}^{H}D_{0^+}^{\gamma,\mu}\left(x\left(t\right)\right) + \frac{\beta}{\sigma^{1-\alpha}}\, {}^{H}D_{0^+}^{\alpha,\mu}\left(x\left(t\right)\right) + kx(t) = J_0 \sum_{n=1}^{N} \delta\left(t - n\tau\right). \tag{19}$$

4.1 Undamped forced oscillator

The solution presented here excludes the damping term in order to understand the effect produced by changing the order of the derivative in the oscillation.

Taking $x(0) = x_0$, $x'(0) = 0$, that is, we assume that the body starts from rest in a predetermined position and denoting $\omega_0^2 = \frac{k}{m}\sigma^{2-\gamma}$ in Eq.(19), we have

$${}^{H}D_{0^+}^{\gamma,\mu}\left(x\left(t\right)\right) + \omega_0^2 x(t) = J_0 \sum_{n=1}^{N} \delta\left(t - n\tau\right).$$

Just apply the Laplace transform on both sides we get,

$$\mathcal{L}\left[{}^{H}D_{0^+}^{\gamma,\mu}\left(x\left(t\right)\right)\right] + \mathcal{L}\left[\omega_0^2 x(t)\right] = \mathcal{L}\left[J_0 \sum_{n=1}^{N} \delta\left(t - n\tau\right)\right],$$

using **Theorem 3.1** we obtain

$$\begin{aligned}
s^{\gamma}X(s) - s^{1+\mu(\gamma-2)}x_0 &= -\omega_0^2 X(s) + J_0 \sum_{n=1}^{N} \mathcal{L}\left[\delta\left(t - n\tau\right)\right] \\
\Leftrightarrow s^{\gamma}X(s) - s^{1+\mu(\gamma-2)}x_0 &= -\omega_0^2 X(s) + J_0 \sum_{n=1}^{N} e^{-n\tau s} \\
\left(s^{\gamma} + \omega_0^2\right) X(s) &= s^{1+\mu(\gamma-2)}x_0 + J_0 \sum_{n=1}^{N} e^{-n\tau s} \\
X(s) &= \frac{s^{1+\mu(\gamma-2)}}{s^{\gamma} + \omega_0^2} + J_0 \sum_{n=1}^{N} \left(\frac{e^{-n\tau s}}{s^{\gamma} + \omega_0^2}\right).
\end{aligned}$$

Finally, it remains to apply the corresponding inverse Laplace transform in separate: the second parcel has a well-known inverse transform (it is the product of the transform by the Heaviside function), while the first parcel has the transform already evaluante in **Theorem 3.1**, so

$$x(t) = x_0 t^{\gamma-\mu(\gamma-2)-2} E_{\gamma,\gamma-\mu(\gamma-2)-1}\left\{-\omega_0^2 t^\gamma\right\} + J_0 \sum_{n=1}^{N} \left[H(t-n\tau)\right](t-n\tau)^{\gamma-1} E_{\gamma,\gamma}\left\{-\omega_0^2 (t-n\tau)^\gamma\right\}. \tag{20}$$

Taking $\mu = 0$ we recover the solution according to Riemann-Liouville and for $\mu = 1$ according to Caputo. Furthermore, for $\gamma = 2$ the two formulations coincide

$$\begin{aligned} x(t) &= x_0 E_{2,1}\left\{-\omega_0^2 t^2\right\} + J_0 \sum_{n=1}^{N} H(t-n\tau)\,(t-n\tau)\,\frac{\sin(\omega_0 (t-n\tau))}{\omega_0 (t-n\tau)} \\ x(t) &= x_0 \cos(\omega_0 t) + J_0 \sum_{n=1}^{N} H(t-n\tau)\frac{\sin(\omega_0 (t-n\tau))}{\omega_0}. \end{aligned} \tag{21}$$

4.2 Graphics

To sketch the graphs of some particular cases, Mathematica will be used. Initially, the solution for $\gamma = 2$. Thus, we plot the graph for the solution of the Eq.(21). Taking the values: $x_0 = 0.01m$, $J_0 = 0.005Ns$, $\tau = 6$, $N = 360$, $\sigma = 1s$, $\omega_0 = 1$ and $t_{\text{total}} = 216$. So,

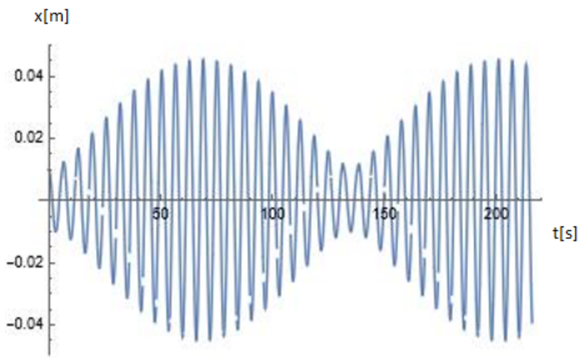

Figure 1: Solution for Eq.(21) with $\gamma = 2$.

The result matches the physical description of the oscillations phenomenon. Despite the oscillator's natural frequency is different from the frequency of the external load, with time it is clear that an overlap of two staggered oscillations occurs.

Now, changing the order $\gamma = 1.9$ follows the outline of the curve.

In both derivative formulations (Figures 2 and 3) for a small value of x_0 an increase in the amplitude of motion occurs, but shortly thereafter there is a relaxation regime that tends to stabilize the amplitude of oscillation.

However, changing $x_0 = 0.5$ increases the relaxation effect.

This solution shows the coupling between the external force, which tends to maintain the beat, and a friction-like effect, which tends to dissipate it.

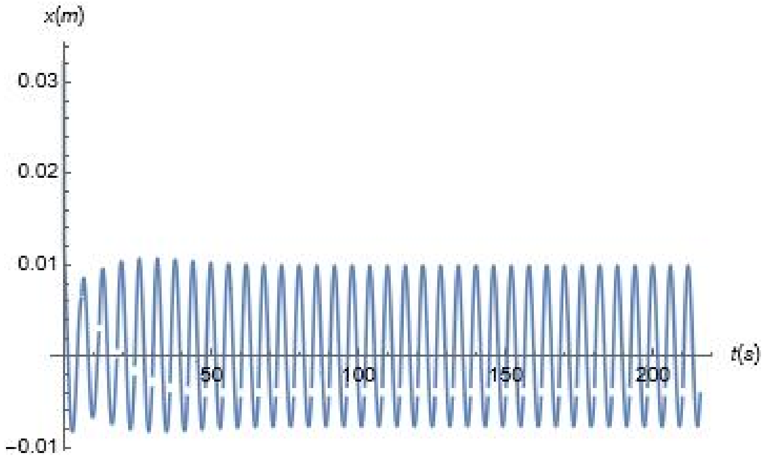

Figure 2: Solution of the Eq.(21) for $\gamma = 1.9$ and $\mu = 0$.

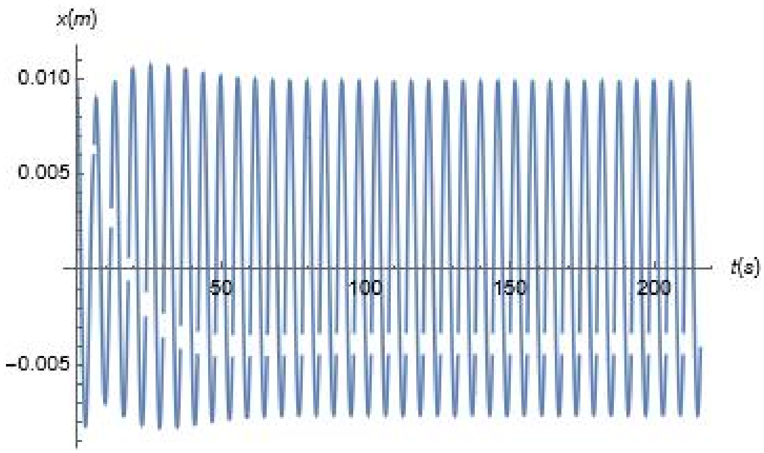

Figure 3: Solution of the Eq.(21) for $\gamma = 1.9$ and $\mu = 1$.

5 Concluding remarks

The solution found modeling the external force by a Dirac comb recovers both for $\mu = 0$ and for $\mu = 1$ when $\gamma = 2$ the same solution: a superposition of oscillations (the first refers to the oscillation of the oscillator itself, while the second refers to the frequency of the external force).

Besides, in all figures there are small “jumps" in the trace of the represented curve, this pattern is maintained due to the action of the Heaviside function.

Finally, in figure 4 when we change the order of the derivative to $\gamma = 1.9$ (value close to 2) the pattern found resembles the smoothing, even if there is no smoothing term β in the original equation. This indicates that the change in the order of the derivative can simulate the dissipative effect on the oscillator, reducing the amplitude of motion and, over a period of time, stabilizing it. This is done by modeling with both formulations the Riemann-Liouville fractional derivative and the Caputo fractional derivative.

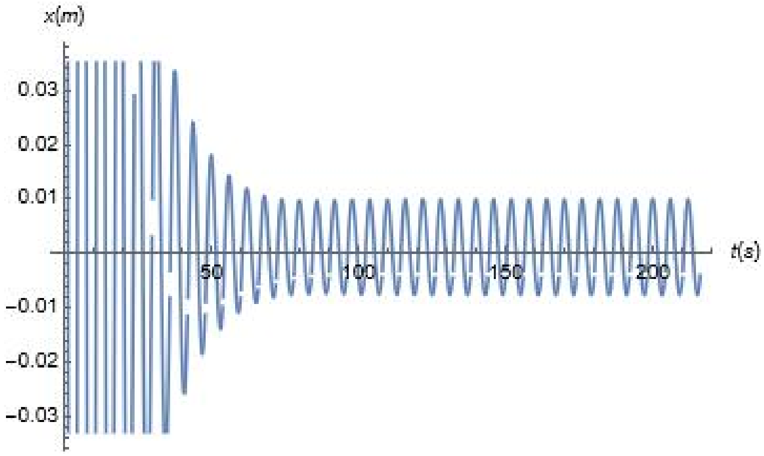

Figure 4: Solution of the Eq.(21) for $\gamma = 1.9$, $\mu = 0$ and $x_0 = 0.5$.

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